Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach

Errata and Clarifications for the Second Edition

Appendix and Index

Updated October 28, 2004

Page 670 In the first sentence after Definition A1.2, $\text{Assoc}(x, y) = (x+y)+z$ should be $\text{Assoc}(x, y, z) = (x + y) + z$.

The words "k-close" were omitted from Definition A1.3, which should read

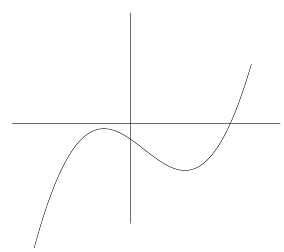
Definition A1.3 (k-close). Two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are k-close if for each $i = 1, \dots, n$, then $\big|[x_i]_k - [y_i]_k\big| \le 10^{-k}$.

Page 671 Exercise A1.2 left out "$\text{Assoc}(x, y, z) =$." The first sentence of the exercise should read

"Show that the functions $A(x, y) = x + y$, $M(x, y) = xy$, $S(x, y) = x - y$, and $\text{Assoc}(x, y, z) = (x + y) + z$ are $\mathbb{D}$-continuous, and that $1/x$ is not."

Page 675 Proposition A2.4: By "exactly" we mean "if and only if." In any case, "if and only if" is more appropriate here. We tend to use "precisely" (or, more rarely, "exactly") when we mean "if and only if" but where the result is fairly obvious, which isn't the case here.

The bottom graph in Figure A2.1 is wrong; it should be:



Page 682 Restatement of Theorem 2.7.13: in the next-to-last line, it should be "has a unique solution in the closed ball $\overline{U_0}$".

Page 691 We corrected Equation 2.9.13 in Section 2.9 (page 270). Of course it should also be corrected here:

$$R_1 = R|L^{-1}|^2 \left(\sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}} - |L| \right). \qquad 2.9.13$$

Page 692 The proof of Theorem 2.9.7 does not include a proof of the last statement, concerning Equation 2.9.13. Here is the missing proof:

Proving Equation 2.9.13

Suppose $|\mathbf{x}-\mathbf{x}_0| < R_1$. Then

$$|\mathbf{f}(\mathbf{x})-\mathbf{f}(\mathbf{x}_0)| \;\le\; |\mathbf{x}-\mathbf{x}_0| \sup|[\mathbf{Df}(\mathbf{x})]| \;\le\; R_1 \sup|[\mathbf{Df}(\mathbf{x})]|. \qquad \text{A7.11}$$

We find a bound for $|[\mathbf{Df}(\mathbf{x})]|$:

$$|[\mathbf{Df}(\mathbf{x})]-[\mathbf{Df}(\mathbf{x}_0)]| = |[\mathbf{Df}(\mathbf{x})]-L| \underbrace{\le}_{\text{Eq. 2.9.11}} \frac{1}{2R|L^{-1}|^2}\,|\mathbf{x}-\mathbf{x}_0| \le \frac{R_1}{2R|L^{-1}|^2}$$

so

$$|[\mathbf{Df}(\mathbf{x})]| \le |L| + \frac{R_1}{2R|L^{-1}|^2}, \quad \text{i.e.,} \quad \sup|[\mathbf{Df}(\mathbf{x})]| = |L| + \frac{R_1}{2R|L^{-1}|^2}. \qquad \text{A7.12}$$

Therefore we want to find the largest R_1 satisfying

$$R \;\ge\; \left(|L| + \frac{R_1}{2R|L^{-1}|^2}\right) R_1. \qquad \text{A7.13}$$

The right-hand side is 0 when $R_1 = 0$ and then increases as R_1 increases, so we want the largest value of R_1 for which the inequality is an equality. Thus we want to solve the quadratic equation

$$R_1^2 + 2R|L^{-1}|^2\,|L|R_1 - 2R^2|L^{-1}|^2 = 0, \qquad \text{A7.14}$$

which gives

$$R_1 = R|L^{-1}|^2 \left(-|L| + \sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}}\right). \qquad \text{A7.15}$$

Appendix A.8 The proof is not as clear as it should be as to why the root found by Newton's method is unique in all of W_0 and not just in U_0. This question is addressed by part (3) of the proof of the *inverse* function theorem, which refers to Remark A5.5 on page 688. Since we treat the implicit function theorem as a special case of the inverse function theorem, this is relevant. In any future editions we plan to put the content of Remark A5.5 in Section 2.7, perhaps immediately after the statement of the Kantorovich theorem.

Page 695 Equation A8.6 is wrong. It should be

$$\mathbf{F}\begin{pmatrix}\mathbf{g}(\mathbf{y})\\ \mathbf{y}\end{pmatrix} = \mathbf{0}.$$

Page 705 Second line after Equation A11.16: it might be clearer to write "which satisfy $|Q^k_{f,\mathbf{a}}(\vec{\mathbf{h}})| \in O(|\vec{\mathbf{h}}|)$", rather than "so that $|Q^k_{f,\mathbf{a}}(\vec{\mathbf{h}})| \in O(|\vec{\mathbf{h}}|)$."

Page 707 Equation A12.3 should end with ds, not dt.

Page 723 In the next-to-last line of the paragraph beginning "Fortunately", the word "volume" should be "measure".

Page 724 Corollary A16.3 is wrong. It is correct if we replace "volume" by "measure." Seeing why the proof is correct requires the following corollary to Theorem 4.4.5:

If f and g are integrable functions on $\mathbb{R}^n$, $g \geq f$, and $\int f(\mathbf{x})|d^n\mathbf{x}| = \int g(\mathbf{x})|d^n\mathbf{x}|$, then $\{\, \mathbf{x} \mid f(\mathbf{x}) \neq g)\mathbf{x})\,\}$ has measure 0.

We propose making this into an exercise, with the hint: Show that if $g(\mathbf{x}_0) > f(\mathbf{x}_0)$ and $g-f$ is continuous at $\mathbf{x}_0$, then $\int g(\mathbf{x})|d^n\mathbf{x}| > \int f(\mathbf{x})|d^n\mathbf{x}|$. Then apply Theorem 4.4.5.

Page 726 Second line of the proof: replacing f by $\chi_X f$ uses the fact that the product of two R-integrable functions is integrable. This is proved in Corollary 4.4.8; it also follows from Theorem 4.3.1. (But the product of two L-integrable functions is not necessarily L-integrable! However, the product of an L-integrable function by a bounded L-integrable function is L-integrable; see the lemma – a somewhat weaker statement – discussed in the note for page 754.)

Page 727 In the first line, we write that every $\mathbf{x}$ is in some paving tile. It is possible that $\mathbf{x}$ may be in more than one tile. By Corollary 4.3.10, such points don't affect integals; however, the definition of $\overline{g}$ should take such points into account:

$$\overline{g}(\mathbf{x}) = \begin{cases} M_{P_{N''}(\mathbf{x})}(f) & \text{if } P_{N''}(\mathbf{x}) \cap \partial\mathcal{D}_N = \phi \text{ and } \mathbf{x} \text{ is contained in a} \\ & \quad \text{single tile} \\ -\sup|f| & \text{otherwise.} \end{cases}$$

We have also rewritten some of the rest of the page, in hopes of making it clearer:

Now we compute the upper sum $U_{\mathcal{P}_{N''}}(f)$, as follows:

$$U_{\mathcal{P}_{N''}}(f) = \sum_{P \in \mathcal{P}_{N''}} M_P(f)\,\mathrm{vol}_n\, P \qquad A17.8$$

$$= \underbrace{\sum_{\substack{P \in \mathcal{P}_{N''},\\ P \cap \partial\mathcal{D}_N = \phi}} M_P(f)\,\mathrm{vol}_n\, P}_{\substack{\text{contribution from } P\\ \text{entirely in dyadic cubes}}} + \underbrace{\sum_{\substack{P \in \mathcal{P}_{N''},\\ P \cap \partial\mathcal{D}_N \neq \phi}} \big(M_P(f)\,\mathrm{vol}_n\, P.}_{\substack{\text{contribution from } P \text{ that intersect}\\ \text{the boundary of dyadic cubes}}}$$

We want a statement that relates integrals computed using dyadic cubes and paving tiles. Since $\sum_{P \in \mathcal{P}} \chi_P = 1$ except on a set of volume 0, The sum of characteristic functions is the constant function 1 except on a set of volume 0.

$$\int_{\mathbb{R}^n} \overline{g}(\mathbf{x})|d^n\mathbf{x}| = \sum_{P\in\mathcal{P}''_N} \int_{\mathbb{R}^n} \overline{g}(\mathbf{x})\chi_P(\mathbf{x})|d^n\mathbf{x}|$$
$$= \sum_{\substack{P\in\mathcal{P}_{N''},\\ P\cap\partial\mathcal{D}_N=\phi}} M_P(f)\,\mathrm{vol}_n P + \sum_{\substack{P\in\mathcal{P}_{N''},\\ P\cap\partial\mathcal{D}_N\neq\phi}} (-\sup|f|)\,\mathrm{vol}_n P.$$

Note that we can write the last term in Equation A17.8 as

$$\sum_{\substack{P\in\mathcal{P}_{N''},\\ P\cap\partial\mathcal{D}_N\neq\phi}} \big(M_P(f)\,\mathrm{vol}_n P = \sum_{\substack{P\in\mathcal{P}_{N''},\\ P\cap\partial\mathcal{D}_N\neq\phi}} \big(M_P(f)\overbrace{-\sup|f|+\sup|f|}^{\text{cancels out}}\big)\,\mathrm{vol}_n P$$
$$= \sum_{\substack{P\in\mathcal{P}_{N''},\\ P\cap\partial\mathcal{D}_N\neq\phi}} \big(-\sup|f|\big)\,\mathrm{vol}_n P + \sum_{\substack{P\in\mathcal{P}_{N''},\\ P\cap\partial\mathcal{D}_N\neq\phi}} \big(M_P(f)+\sup|f|\big)\,\mathrm{vol}_n P.$$

Since $M_P(f)$ is the least upper bound over P while $\sup|f|$ is the least upper bound over $\mathbb{R}^n$, we have $M_P(f)+\sup|f| \leq 2\sup|f|$.

So we can rewrite Equation A17.8 as

$$U_{\mathcal{P}_{N''}}(f) = \int_{\mathbb{R}^n} \overline{g}|d^n\mathbf{x}| + \sum_{\substack{P\in\mathcal{P}_{N''},\\ P\cap\partial\mathcal{D}_N\neq\phi}} \overbrace{\big(M_P(f)+\sup|f|\big)}^{\leq 2\sup|f| \text{ (see note in margin)}}\ \mathrm{vol}_n P. \qquad \text{A17.11}$$

Pages 742, 743, 745 Each page has "integrable functions" that should be "R-integrable functions":

Proposition A21.1: "Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be a sequence of R-integrable functions . . . "

Corollary A21.2: "Let h_k be a sequence of R-integrable nonnegative functions on Q . . . "

Proposition A21.3: "Suppose f_k is a sequence of R-integrable functions all satisfying . . . "

(These statements come in in the course of proving Theorem 4.11.4, which is a statement about Riemann integrals.)

Page 744 The last sum in Equation A21.8 should have i, not k:

$$\sum_{i=1}^{\infty} \int h_i|d^n\mathbf{x}|.$$

Page 747 The letter A is is in the wrong font in one place (A should be A):

"But this argument requires "measure 0." To apply it to the case where $A \neq 0$ . . . "

Page 748 Equation A21.26: In the first line, the sums should be over $C \subset Y_0$, not $C \in Y_0$. But then we also have to specify that the C are in $\mathcal{D}_{N_0}(\mathbb{R}^n)$. This gives

$$\begin{aligned}\mathrm{vol}_n(Y_0)\frac{A}{\epsilon} = \sum_{\substack{C\subset Y_0\\ C\in\mathcal{D}_{N_0}(\mathbb{R}^n)}} \frac{A}{\epsilon}\,\mathrm{vol}_n(C) \;&\le \sum_{\substack{C\subset Y_0\\ C\in\mathcal{D}_{N_0}(\mathbb{R}^n)}} M_C(h_0)\,\mathrm{vol}_n(C)\\ &\le \sum_{C\in\mathcal{D}_{N_0}} M_C(h_0)\,\mathrm{vol}_n(C) = U_{N_0}(h_0) \le 2A,\end{aligned} \qquad \text{A21.26}$$

In Equation A21.29, h_m comes with a $+$ sign and h_{m+1} with a $-$ sign; it should be reversed. In the second line, the $=$ should be $<$. So the equation should read

$$\begin{aligned}\int_{\mathbb{R}^n} g_{m+1}(\mathbf{x})|d^n\mathbf{x}| &= \left(\int_{\mathbb{R}^n} h_{m+1}(\mathbf{x})\,|d^n\mathbf{x}| - A\right) - \left(\int_{\mathbb{R}^n} h_m(\mathbf{x})\,|d^n\mathbf{x}| - A\right)\\ &\le \frac{A}{4^{m+3}} + \frac{A}{4^{m+2}} < \frac{A}{2\cdot 4^{m+1}}.\end{aligned} \qquad \text{A21.29}$$

Page 749 Equation A21.35 : on the far right, the A in the numerator should be ϵ:

$$\mathrm{vol}_n(Y_{m+1}) \le \epsilon\frac{2^{m+1}}{4^{m+1}} = \frac{\epsilon}{2^{m+1}}.$$

Page 750 Equation A21.39: Writing "for $j = 2, \dots, \infty$" is fairly standard but it would be better as "$2 \le j < \infty$"; we do not mean to suggest that $j = \infty$.

The equation in the footnote contains mistakes with the absolute value signs and parentheses. It should be:

$$\begin{aligned}\int_{\mathbb{R}^n} |g_{k,1}(\mathbf{x})||d^n\mathbf{x}| &= \int_{\mathbb{R}^n} \left|\sum_{i=1}^{\infty} f_{k,i}(\mathbf{x}) - \sum_{i=m(k)+1}^{\infty} f_{k,i}(\mathbf{x})\right| |d^n\mathbf{x}|\\ &\le \int_{\mathbb{R}^n} \left|\sum_{i=1}^{\infty} f_{k,i}(\mathbf{x})\right||d^n\mathbf{x}| + \sum_{i=m(k)+1}^{\infty} \int |f_{k,i}(\mathbf{x})|\,|d^n\mathbf{x}|\\ &\le \int_{\mathbb{R}^n} |f_k(\mathbf{x})|d^n\mathbf{x}| + \frac{1}{2^k}.\end{aligned}$$

Page 753 Two lines after Equation A21.49 replace "volume 0" by "measure 0" in two places. This uses Corollary A16.3, which has been corrected (it concerns measure, not volume).

Sentence right after Equation A21.50: third and fourth "equalities", not "inequalities."

Elaboration: In line two of the proof, we are using Fubini for Riemann integrals. More precisely, Equation A21.49 is true for Riemann integrals if one ignores sets of measure 0, and so it is true without restriction for Lebesgue integrals.

Page 754 Third displayed equation: the bracket on the left should say "finite because f, hence g, is L-integrable."

In the paragraph beginning "For the converse", $\mathbb{R}^n$ should be $\mathbb{R}^{n+m}$ – i.e., "every closed cube $C \in \mathcal{D}_0(\mathbb{R}^n)$ is compact" should be

"every closed cube $C \in \mathcal{D}_0(\mathbb{R}^{n+m})$ is compact."

Even with that correction, we were not quite rigorous in arguing that $f\chi_C$ is L-integrable. Here is another version:

Lemma *If f is L-integrable on $\mathbb{R}^n$, and g is an R-integrable function with $0 \le g \le 1$, then fg is L-integrable.*

Proof. Since f is L-integrable, we can set $f = \sum_k f_k$ with the f_k R-integrable and

$$\sum_k \int_{\mathbb{R}^n} |f_k(\mathbf{x})|\,|d^n\mathbf{x}| < \infty.$$

We have $fg = \sum_k f_k g$, where $f_k g$ is R-integrable; since $0 \le g \le 1$, we have

$$\sum_k \int_{\mathbb{R}^n} |f_k g(\mathbf{x})|\,|d^n\mathbf{x}| \le \sum_k \int_{\mathbb{R}^n} |f_k(\mathbf{x})|\,|d^n\mathbf{x}| < \infty.$$

Therefore fg is L-integrable. $\square$

Now take a closed cube $C \in \mathcal{D}_0(\mathbb{R}^n)$ and cover it by finitely many balls $B_1, B_2, \ldots, B_N$, over which (by the first hypothesis of the converse) f is L-integrable. Then we can write

$$f\chi_C = f\chi_C\chi_{B_1} + f\chi_C\chi_{B_2-\cup(B_2-B1)} + \cdots + f\chi_C\chi_{B_N - \cup_{j=1}^{N-1} B_j}$$

By the above lemma, the terms on the right are all L-integrable, so by Proposition 4.11.15, $f\chi_C$ is L-integrable.

Page 755 We claim we are proving the "if" part in the text, leaving "if only" as an exercise. Actually, it's the reverse. It would be clearer to use $\Longrightarrow$ and $\Longleftarrow$. In the text we prove ($\Longrightarrow$) (that if f is L-integrable, then $|\det[\mathbf{D}\Phi]|(f\circ\Phi)$ is L-integrable and the formula is correct).

Margin note: The first and third equalities of Equation A21.59 are applications of Theorem 4.11.16. In both cases, the hypothesis of that theorem is satisfied by Equation A21.58. We could add an extra step:

$$\int_V f(\mathbf{v})|d^n\mathbf{v}| \overbrace{=}^{\text{Eq.A21.57}} \int_V \sum_{k,i} f_{k,i}(\mathbf{v})|d^n\mathbf{v}| \overbrace{=}^{\text{Thm. 4.11.16}} \sum_{k,i}\int_V f_{k,i}(\mathbf{v})|d^n\mathbf{v}|$$

$$\overbrace{=}^{\text{Thm. 4.10.12}} \sum_{k,i}\int_U \big|\det[\mathbf{D}\Phi(\mathbf{u})]\big|\; f_{k,i}(\Phi(\mathbf{u}))\;|d^n\mathbf{u}|$$

$$\overbrace{=}^{\text{Thm. 4.11.16}} \int_U \big|\det[\mathbf{D}\Phi(\mathbf{u})]\big|\Big(\sum_{k,i} f_{k,i}(\Phi(\mathbf{u}))\Big)|d^n\mathbf{u}|; \qquad \text{A21.59}$$

Page 755 Last line: "Exercise A21.2" should be "Exercise A21.5."

Page 759 Exercise A21.5 (last exercise of the section, incorrectly denoted A21.2): as indicated in the note for page 755, this exercise asks you to prove the "if" part, not "only if". In future editions this exercise will be:

Justify the ($\Longleftarrow$) part of Theorem 4.11.20 (if $|\det[\mathbf{D}\Phi]|(f\circ\Phi)$ is L-integrable, then f is L-integrable and the formula given in the theorem is correct), using the ($\Longrightarrow$) part and the chain rule.

Page 760 In Figure A22.1, the top lines in both rectangles should be darker.

Page 765 The margin note should start with "In," not "in."

Page 763 4th line: rather than state that the exterior derivative $d\varphi$ is a $(k+1)$-form, we should say "Since φ is a k-form, the exterior derivative $d\varphi$ should be a $(k+1)$-form. Thus we need to evaluate it on $k+1$ vectors and check that it is multilinear and alternating. This involves integrating φ ... "

Page 766 Definition A24.1 should read

Definition A24.1 (Pullback by a linear transformation). Let V, W be vector spaces, and $T: V \to W$ be a linear transformation. Then T^* is a linear transformation $A^k(W) \to A^k(V)$, defined as follows: if φ is a k-form on W, then $T^*\varphi$ is the k-form on V given by

$$T^*\varphi(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k) = \varphi\big(T(\vec{\mathbf{v}}_1),\dots,T(\vec{\mathbf{v}}_k)\big). \qquad \text{A24.1}$$

Page 767 In the last margin note, an end parenthesis is missing: $g(P_{\mathbf{f}(\mathbf{x})}$ should be $g(P_{\mathbf{f}(\mathbf{x})})$

Page 767 In the line immediately before Definition A24.4 there is a superfluous comma.

Page 769 In the last line of Equation A24.14, the $\mathbf{g}^*\mathbf{f}^*$ should be $\mathbf{f}^*\mathbf{g}^*$:

$$= \mathbf{g}^*\varphi\Big(P_{\mathbf{f}(\mathbf{x})}\big([\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_1, \dots, [\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_k\big)\Big) = \mathbf{f}^*\mathbf{g}^*\varphi\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)\big)$$

Page 770 We have rewritten the first paragraph:

Why does this result matter? To define the exterior derivative, we used the parallelograms $P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$. To do this, we had to know how to draw straight lines from one point to another; we were using the linear (straight) structure of a vector space. (We used $\mathbb{R}^n$, but any vector space would have done.) Theorem A24.8 says that "curved parallelograms" (little bits of manifolds) would have worked as well. Thus the exterior derivative is not restricted to forms defined on vector spaces.

(In this book we have discussed forms on vector spaces, but differential forms can also be defined on manifolds embedded in $\mathbb{R}^n$ and on abstract manifolds. Theorem A24.8 says that an exterior derivative exists for such forms. It is a crucial result, since forms without an exterior derivative would be of no interest.)

Title of Theorem A24.8: By "intrinsic" we mean "inherent: independent of some external conditions or circumstances." The pullback of a form by a C^1 mapping is a C^1 change of variables. Equation A24.17 says that when a form is pulled back by a C^1 mapping, its exterior derivative remains the same, translated appropriately into the new variables.

Page 770 In the first line of Equation A24.18 (last term), $[\mathbf{D}g(\mathbf{x})]$ should be $[\mathbf{Df}(\mathbf{x})]$:

$$\begin{aligned} \mathbf{f}^* dg\big(P_{\mathbf{x}}(\vec{\mathbf{v}})\big) &= dg\big(P_{\mathbf{f}(\mathbf{x})}[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}\big) = [\mathbf{D}g(\mathbf{f}(\mathbf{x}))][\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}} \\ &= [\mathbf{D}g \circ \mathbf{f}(\mathbf{x})]\vec{\mathbf{v}} = d(g \circ \mathbf{f})\big(P_{\mathbf{x}}(\vec{\mathbf{v}})\big) = d(\mathbf{f}^* g)\big(P_{\mathbf{x}}(\vec{\mathbf{v}})\big). \end{aligned} \qquad \text{A24.18}$$

Page 770 Equation A24.19: above the first equal sign, "Theorem A6.7.8" should be "Theorem 6.7.8."

Inside back cover The "useful formulas: trigonometry" would be more useful if they were all correct! Sorry! The fourth and fifth formulas should be

$$\cos\alpha = \sin(\pi/2 - \alpha) \quad \text{and} \quad \sin\alpha = \cos(\pi/2 - \alpha).$$

(For the formula for sine, it doesn't make a difference, but for cosine, it does.)

Page 782 In Exercise A25.2, "(proof of Lemma A25.12)" should be "(see Equation A25.12)".

Index

Page 792 dominated convergence (Lebesgue), 515 (not 516)

The listing for *diffeomorphism* on page 514 should be deleted.

Page 797 triangle inequality, 76–77 (not 76)

Additions to index: