Errata and Clarifications for the Second Edition: Chapter 6

updated April 14, 2004

Page 561 Definition 6.1.3. "An elementary k-form", not "A elementary k-form".

Page 562 The righthand side of Equation 6.1.14 should be

$$\sum_{i=1}^{k-1} a_i \varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{k-1}, \vec{\mathbf{v}}_i).$$

The first term is $a_1\phi(\vec{\mathbf{v}}_1, ..., \vec{\mathbf{v}}_{k-1}, \vec{\mathbf{v}}_1)$, the second is $a_2\phi(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_{k-1}, \vec{\mathbf{v}}_2)$, and so on.

Page 563 Clarification for Example 6.1.8:

The function $W_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$ is a 1-form on $\mathbb{R}^n$ because it is a function of one vector and it is linear as a function of $\vec{\mathbf{w}}$. The requirement that it be antisymmetric is automatically satisfied, since it is a function of only one vector.

Page 564 Equation 6.1.23 should be

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\vec{\mathbf{e}}_{j_1}, \dots \vec{\mathbf{e}}_{j_k}). \qquad 6.1.23$$

Equation 6.1.24 should be

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k}(\vec{\mathbf{e}}_{j_1}, \dots, \vec{\mathbf{e}}_{j_k}) = 1. \qquad 6.1.24$$

Page 568 Not an error, but in subsequent editions we plan to add the following to the first margin note:

If V is k-dimensional, a nonzero element of $A^k(V)$ will correspond, via $\Phi_{\{\underline{\mathbf{b}}\}}$ as in Equation 6.1.30, to a nonzero multiple of det $\in A^k(\mathbb{R}^k)$. In particular, a nonzero element of $A^k(V)$ evaluated on k linearly independent vectors always returns a nonzero number.

Page 569 The last margin note refers to nonexistent parts a) and b) of Definition 6.1.1. That sentence should read

The wedge product $\varphi \wedge \omega$ satisfies the requirements of Definition 6.1.1 for a form (multilinearity and antisymmetry).

Page 570 Discussion after Definition 6.1.22:

We will assume that these functions are of class at least C^2: we will need C^1 to define the exterior derivative and C^2 for Theorem 6.7.7 to be true.

Page 571 Exercise 6.1.2 (a): $dx_3 \wedge dx_2 \wedge x_4$ should be $dx_3 \wedge dx_2 \wedge dx_4$.

Page 580 Caption to Figure 6.3.1: "we choose a tangent vector field", not "we choose tangent vector field".

Page 580 After Definition 6.3.1, add

If (M, ω) is a manifold oriented by the form ω, then $-(M, \omega)$ will refer to M with the opposite orientation. It follows that $-(M, \omega) = (M, -\omega)$.

Page 581 Proposition 6.3.5: As written, this proposition assumes that an appropriate normal vector field can be chosen. Of course, that is not always the case, as is clear from considering the Moebius strip. The proposition should read

Proposition 6.3.5 (Orienting a surface in $\mathbb{R}^3$). *Let $S \subset \mathbb{R}^3$ be a smooth surface. In this case $T_{\mathbf{x}}S$ is two-dimensional, and an element of the line $A^2(T_{\mathbf{x}}S)$ is a 2-form. Suppose there exists a normal vector field $\vec{\mathbf{n}}$, as shown in Figure 6.3.2: for each $\mathbf{x} \in S$ we can choose a nonzero vector $\vec{\mathbf{n}}(\mathbf{x}) \in T_{\mathbf{x}}S^{\perp}$, such that $\vec{\mathbf{n}}(\mathbf{x})$ varies continuously with $\mathbf{x}$. Then S can be oriented by the 2-form field $\omega_{\mathbf{x}} \in A^2(T_{\mathbf{x}}S)$ given by*

$$\omega_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2) = \det[\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2], \quad \text{where } \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 \in T_{\mathbf{x}}S. \qquad 6.3.4$$

In the proof, we should write "$\omega_{\mathbf{x}}$ is not the zero element of $A^2(T_{\mathbf{x}}S)$," not "$\omega_{\mathbf{x}}$ is not the 0-form":

Proof. The 2-form $\omega_{\mathbf{x}}$ is not the zero element of $A^2(T_{\mathbf{x}}S)$, since if $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are linearly independent and are in $T_{\mathbf{x}}S$, then $\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are linearly independent, with nonzero determinant; $\omega_{\mathbf{x}}$ varies continuously because $\det[\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2]$ is a polynomial, and (Corollary 1.5.30) polynomial functions are continuous. □

Page 582 Proposition 6.3.8: We should have said "Suppose there exists a normal vector field $\vec{\mathbf{n}}$", not "Choose a normal vector field $\vec{\mathbf{n}}$". If no normal vector field $\vec{\mathbf{n}}$ exists, then the manifold is not orientable.

Page 583 In the second line of proof of Proposition 6.3.9, an end parenthesis is missing: $A^0(\{\vec{\mathbf{0}}\}) = \mathbb{R}$, not $A^0(\{\vec{\mathbf{0}}\} = \mathbb{R}$.

Page 584 In Equation 6.3.9, the second equality is incorrect; the second determinant is opposite the first. The discussion should read:

... so we are looking for either

$$\omega_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \det \begin{bmatrix} y & 0 & v_1 & w_1 \\ x & 2y & v_2 & w_2 \\ w & 2z & v_3 & w_3 \\ z & 0 & v_4 & w_4 \end{bmatrix} \quad \text{or} \quad \omega'_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \det \begin{bmatrix} 0 & y & v_1 & w_1 \\ 2y & x & v_2 & w_2 \\ 2z & w & v_3 & w_3 \\ 0 & z & v_4 & w_4 \end{bmatrix}. \qquad 6.3.9$$

These 2-forms are nonzero elements of $A^2(T_{\mathbf{x}}S)$, i.e., $\omega_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = -\omega'_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}}) \neq 0$ if $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in T_{\mathbf{x}}S$ are linearly independent. The first gives

$$\begin{aligned} \omega_{\mathbf{x}} = -2z^2\, dx \wedge dy + 2yz\, dx \wedge dz + (2xz - 2yw)\, dx \wedge dw \\ + 2y^2\, dz \wedge dw - 2zy\, dy \wedge dw. \end{aligned} \qquad 6.3.10$$

Page 584 Footnote: The footnote is not well written. It should be replaced by

"A nonzero k-form on a k-dimensional vector space returns 0 when evaluated on k vectors if and only if the vectors are linearly dependent."

Page 589 Part (c) of Exercise 6.3.12: The notation is inconsistent. We will change $\mathbf{v}_1$ to $\mathbf{v}$ and $\mathbf{v}_2$ to $\mathbf{w}$:

(c) Show that given any two linearly independent vectors $\mathbf{u}_1, \mathbf{u}_2$ in $\mathbb{R}^n$, $n > 2$, there exist maps $\mathbf{v}, \mathbf{w} : [0,1] \to \mathbb{R}^n$ such that

$$\mathbf{v}(0) = \mathbf{u}_1,\ \mathbf{v}(1) = \mathbf{u}_2,\ \mathbf{w}(0) = \mathbf{u}_2,\ \mathbf{w}(1) = \mathbf{u}_1,$$

and for each t, $\mathbf{v}(t)$ and $\mathbf{w}(t)$ are linearly independent.

Page 590 Exercise 6.3.17, part (b): The curve C should be smooth.

Page 591 Third margin note: Definition 6.4.2, not 6.4.1.

Page 592 We forgot to put a $\triangle$ to mark the end of Example 6.4.3.

Page 592 Footnote: "It is never the 0-form" should be "it is never the zero element of $A^2(T_{\mathbf{x}}S)$."

Page 595 Second line in margin: pullback of ω, not pullback of φ.

Margin note half-way down the page: Equation 6.4.20, not 6.4.19.

Page 596 First margin note, third line: there is an extra colon.

Page 602 The solution to Exercise 6.4.6 uses the formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Justifying this formula uses three statements taught in one-variable calculus and the fact (Proposition 1.5.34) that absolute convergence implies convergence. The three statements are the expression of $\sin t$, $\cos t$, and e^t, for t real, in terms of power series:

$$
\begin{aligned}
\sin t &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \\
\cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \\
e^t &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots . \qquad (1)
\end{aligned}
$$

First, let us show that for a complex number z, we can define e^z by the power series

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots .$$

We know it is true in the special case where z is real. We need to check that the series converges. The series $1 + |z| + \left|\frac{z^2}{2!}\right| + \cdots$ converges, since (by Equation (1): $|z|$ is a real number)

$$\sum_{k=0}^{\infty} \left| \frac{z^k}{k!} \right| = \sum_{k=0}^{\infty} \frac{|z|^k}{k!} = e^{|z|}$$

converges. So Proposition 1.5.34 says that $\displaystyle\sum_{k=0}^{\infty} \frac{z^k}{k!}$ converges.

Now write

$$
\begin{aligned}
\cos t + i \sin t &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \right) \\
&= \left(1 + \frac{(it)^2}{2!} + \frac{(it)^4}{4!} + \frac{(it)^6}{6!} + \cdots\right) + \left(it + \frac{(it)^3}{3!} + \frac{(it)^5}{5!} + \right) \\
&= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots = e^{it}.
\end{aligned}
$$

Page 603 Caption for Figure 6.5.2: In two places (the first line and immediately after the displayed equation), $x\,dx + y\,dx$ should be $x\,dx + y\,dy$.

Page 603 The sentence "the requirement of antisymmetry then says that $f(-P_{\mathbf{x}}) = -f(\mathbf{x})$" should be deleted.

Page 605 Figure 6.5.7: the vector field should be turning clockwise, as shown below.

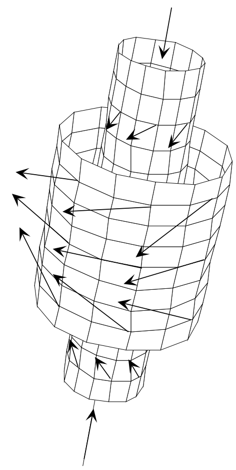

FIGURE 6.5.4. Corrected figure

Page 606 Line 4: clockwise, not counter-clockwise.

Page 607 There were two mistakes in Example 6.5.6 The tangent vector field is $\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$, not $\begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix}$, and $\vec{\gamma}'(t)$ is $\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$. Thus the first half of the example should read:

What is the work of the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}$ over the helix oriented by the tangent vector field $\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$, and parametrized by $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$, for $0 < t < 4\pi$?

The parametrization preserves orientation, since

$$\omega\big(\vec{\gamma}'(t)\big) = \underbrace{\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}}_{\vec{\mathbf{t}}(t)} \cdot \underbrace{\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}}_{\vec{\gamma}'(t)} = 2 > 0. \qquad 6.5.13$$

Page 607 Immediately before Equation 6.5.15: “orientation-preserving”, not “orientation-preseving”.

Page 609 Last margin note: the signs are reversed in the matrix; it should be $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Page 616 Definition 6.6.2, part (2): $\left[\mathbf{D}\begin{pmatrix}\mathbf{f}\\ g\end{pmatrix}\left(\mathbf{x}\right)\right]$, not $\left[\mathbf{D}\frac{\mathbf{f}}{g}\left(\mathbf{x}\right)\right]$

Page 618 Immediately before Example 6.6.6 we have added

An *oriented* piece-with-boundary of a manifold is a piece of an oriented manifold: the piece inherits the orientation of the manifold. Given $X \subset (M, \omega)$, we write $-X$ to denote X as a subset of $-M$.

Page 619 Caption to Figure 6.6.7, last sentence:
"However, the two-dimensional...", not "However, that the two-dimensional...".

Equation 6.6.5 has a misplaced end parenthesis; the first equation should be

$$g(\mathbf{y}) = (\mathbf{y} - \mathbf{0}) \cdot \vec{\mathbf{w}}_i = 0.$$

Page 619 Example 6.6.7: We changed the second paragraph to read

Let $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ be linearly independent vectors in $\mathbb{R}^n$. We will show that the parallelogram $P_{\mathbf{0}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is a piece-with-boundary of the subspace $M \subset \mathbb{R}^n$ spanned by those vectors, i.e., $M = \operatorname{Sp}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$.

In doing so we removed the part about $\mathbf{f}$, which we put into the fourth paragraph:

First we will show that any point that is in a face and is not in any edge is a smooth point. Choose vectors $\vec{\mathbf{w}}_1, \dots, \vec{\mathbf{w}}_k$ in M so that $\vec{\mathbf{w}}_i$ is orthogonal to $\vec{\mathbf{v}}_1, \dots, \widehat{\vec{\mathbf{v}}_i}, \dots \vec{\mathbf{v}}_k$; change the sign if necessary, so that $\vec{\mathbf{w}}_i \cdot \vec{\mathbf{v}}_i > 0$. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be a linear transformation whose kernel is precisely M; note that $\mathbf{f}$ is necessarily surjective. Then $P_{\mathbf{0}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is defined by the equalities and inequalities

Page 622 Last line of Definition 6.6.10: $\partial_1 P$ should be $\partial_M P$.

Page 622 Notational inconsistency. We use both ω^∂ and ω_∂ for the form orienting the boundary. In future printings we will stick with ω^∂.

Page 623 Equation 6.6.16 has a superfluous end parenthesis; it should be

$$\omega^\partial_{\mathbf{x}}(\vec{\mathbf{v}}) = \det\big(\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_{\text{out}}, \vec{\mathbf{v}}\big).$$

Page 623 The first four lines of the new subsection now read

We saw earlier that an oriented k-parallelogram $P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is a piece-with-boundary of $\operatorname{Sp}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ when those vectors are linearly independent.

Since Sp $(\vec{\mathbf{v}}_1, \ldots, \vec{\mathbf{v}}_k)$ is oriented by the order of the vectors, a k-parallelogram is an *oriented* piece-with-boundary. As such its boundary carries an orientation.

In addition, we added this as a margin note:

Recall (Proposition 6.3.9) that a 0-dimensional manifold is oriented by the choice of sign. Thus an oriented 0-parallelogram $P_{\mathbf{x}}$ is either $+P_{\mathbf{x}}$ or $-P_{\mathbf{x}}$. (Recall from the remark immediately after Definition 6.3.13 that the description of orientation in terms of direct bases does not work in the 0-dimensional case.) Since $P_{\mathbf{x}}$ is itself a manifold, its boundary is empty, which is what Proposition 6.6.15 says when $k = 0$.

Page 626 Exercise 6.6.1: The way this exercise was stated in the first printing was not optimal; it should say:

"Use Definition 5.2.1 to show that a single point in any $\mathbb{R}^n$ never has 0-dimensional volume 0."

Page 626 Exercise 6.6.5: $\vec{\nabla}$ denotes the transpose of the derivative:

$$\vec{\nabla} f(\mathbf{x}) = [\mathbf{D}f(\mathbf{x})]^\top.$$

For a function $f : \mathbb{R}^n \to \mathbb{R}$, $\vec{\nabla} f(\mathbf{x})$ is a vector whereas $[\mathbf{D}f(\mathbf{x})]$ is a line matrix.

Note that $\vec{\nabla} f(\mathbf{x}_0)$ is orthogonal at $\mathbf{x}_0$ to the manifold X of equation $f(\mathbf{x}) = 0$: since $T_{\mathbf{x}_0}X = \ker[\mathbf{D}f(\mathbf{x}_0)]$, if $\vec{\mathbf{v}} \in T_{\mathbf{x}_0}X$, then

$$\vec{\nabla} f(\mathbf{x}_0) \cdot \vec{\mathbf{v}} = [\mathbf{D}f(\mathbf{x}_0)]\vec{\mathbf{v}} = 0.$$

Page 627 Exercise 6.6.8 should say that M is oriented by $dx_1 \wedge dx_2 \wedge dx_3$.

Page 629 Last line of Remark 6.7.2: "in higher dimensions", not "to ... ".

Page 631 In the first line of Equation 6.7.14, φ should be ψ.

Page 632 In Theorem 6.7.7, we should have said, "For any k-form φ of class C^2 ... ."

Page 634 Exercise 6.7.6: "Compute the following exterior derivatives," not "Compute the exterior following derivatives."

Exercise 6.7.7: In part (b), "check the computation in (b)" should be "check the computation in (a)."

Exercise 6.7.10: "face" rather than "edge" in two places.

Page 635 The formulas for the gradient and the divergence work in any $\mathbb{R}^n$, but there is no obvious generalization of the curl, other than the exterior derivative.

Page 636 The last term on the right-hand side of Equation 6.8.5 should be $D_3 f v_3$, not $D_3 v_3$.

Page 639 The geometric interpretation of the curl that is given applies equally to $\operatorname{curl} F$ and $-\operatorname{curl} F$. It should read:

The curl probe. Consider an axis, free to rotate in a bearing that you hold, and having paddles attached, as in Figure 6.8.2. If you stand this paddle wheel on a table, paddle end down, next to a clock lying flat on the table, then the wheel turns clockwise if it follows the motion of the hands of the clock. We will orient the axis of the probe up, away from the paddle. We will assume that the bearing is packed with a viscous fluid, so that its angular speed (not acceleration) is proportional to the torque exerted by the paddles. If a fluid is in constant motion with velocity vector field $\vec{F}$, then the curl of the velocity vector field at $\mathbf{x}$, $(\vec{\nabla} \times \vec{F})(\mathbf{x})$, is measured as follows:

> *Insert the paddle of the curl probe into the vector field at a point* $\mathbf{x}$ *and adjust it so that it is spinning counterclockwise the fastest. Then the curl of the vector field at* $\mathbf{x}$ *points in the direction of axis of the probe. The speed at which the probe spins is proportional to the magnitude of the curl.*

Page 640 In the margin note, curl $\vec{F}$ should be curl curl $\vec{F}$. In $\mathbb{R}^3$ the Laplacian is often denoted Δ. Note that Δ is the dot product $\nabla \cdot \nabla$:

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \cdot \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = D_1^2 + D_2^2 + D_3^2.$$

Thus Δ is sometimes denoted ∇^2.

Page 642 We omitted part (c) of Exercise 6.8.10:

(c) Compute it again, directly from the definition of the exterior derivative.

Page 642 Part (c) of Exercise 6.8.11 was not clearly stated. We mean that you should compute them directly from the definition of the exterior derivative. We strongly recommend doing at least part of part (c).

Page 650 Exercise 6.9.6: We should have specified $a, b > 0$ and we should have discussed orientation. Future editions will contain a new part (b):

(b) Show that $(x_1\, dx_2 - x_2\, dx_1) \wedge (x_3\, dx_4 - x_4\, dx_3)$ is an orientation of the surface. Does your parametrization preserve or reverse orientation?

The current parts (b), (c), and (d) will become (c), (d), and (e).

Page 651 Theorem 6.10.2: The reference to Definition 6.6.13 should be to Definition 6.6.10.

Page 657 Exercise 6.10.8 is wrong as written; indeed, it contradicts Exercise 6.10.7. The vector fields should be $\begin{bmatrix} xy^2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -x^2y \end{bmatrix}$.

Page 658 Exercise 6.10.15, part (b): "the surface $X_{p,q}$ of equation $z_1^p + z_2^q$ should be "the surface $X_{p,q}$ of equation $z_1^p + z_2^q = 0$."

Page 659 We should have chosen our bicycle trip at the top of the hill; then it would be clear that if a cyclist starts and ends at the same point, he or she does no work against gravity. In the absence of friction (including friction from braking) a cyclist could zoom down one hill and coast back up the next, without doing any work.

Page 661 Margin note: Equation 6.5.12, not 5.6.1.

Page 662 The function described in Theorem 6.11.5 is unique up to the addition of an arbitrary constant. Thus the function given in Equation 6.11.24 is not the only potential of the vector field; any function $\frac{xy^2}{2} + xyz + c$, where c is an arbitrary constant, is also a potential of $\vec{F}$.

Page 664 Exercise 6.1.3, part (b): "Sketch the potential" should be "sketch the electric field."

Page 665 Exercise 6.11: "for the following 1-forms on $\mathbb{R}^2$ should be "for the following 1-forms."

Page 666 Exercise 6.12: the matrix should be $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This affects parts (a) and (b).

Page 667 In Exercise 6.18, part (b), the displayed equation should be

$$\mathrm{vol}_n\big(B_1^n(\mathbf{0})\big) = \frac{1}{n}\,\mathrm{vol}_{n-1}(S^{n-1}).$$