Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach

Errata and Clarifications for the Second Edition

Chapter 4

updated October 28, 2004

Page 402 Six lines from the bottom, in the statement labeled (2): exists, not exits.

Page 411 We should have included in this section the following statement about how volume scales, or an arbitrary subset of $\mathbb{R}^n$:

Proposition (Scaling volume). *If $A \subset \mathbb{R}^n$ has volume and $t \in \mathbb{R}$, then tA has volume and* $\mathrm{vol}_n(tA) = t^n \,\mathrm{vol}_n(A)$.

Proof. By Proposition 4.1.19, this is true if A is a parallelogram, in particular if A is a cube $C \in \mathcal{D}_N$. Assume A is any subset of $\mathbb{R}^n$. For any N, let f_N be the function that is the constant function 1 on cubes in $\mathcal{D}_N$ that are completely inside A, and let g_N be the function that is the constant function 1 on cubes in $\mathcal{D}_N$ that completely cover A:

$$f_N = \sum_{\substack{C \in \mathcal{D}_N, \\ C \subset A}} \chi_C, \quad g_N = \sum_{\substack{C \in \mathcal{D}_N, \\ C \cap A \neq \emptyset}} \chi_C,$$

so that $f_N \le \chi_A \le g_N$. Then

$$f_N(t\mathbf{x}) \le \chi_A(t\mathbf{x}) = \chi_{tA}(\mathbf{x}) \le g_N(t\mathbf{x}).$$

By Proposition 4.1.19,

$$\int f_N(t\mathbf{x}) = \sum_{\substack{C \in \mathcal{D}_N, \\ C \subset A}} \int \chi_C(t\mathbf{x}) = \sum_{\substack{C \in \mathcal{D}_N, \\ C \subset A}} \overbrace{\int \chi_{tC}(\mathbf{x})}^{\mathrm{vol}_n\, tC} \underbrace{=}_{\text{Prop.4.1.19}} t^n \sum_{\substack{C \in \mathcal{D}_N, \\ C \subset A}} \overbrace{\int \chi_C(\mathbf{x})}^{\mathrm{vol}_n\, C} = t^n \int f_N(\mathbf{x}).$$

(We omitted the $|d^n\mathbf{x}|$ in the integrals above in hopes of making the equation more readable.) Similarly, $\int g_N(t\mathbf{x}) = t^n \int g_N(\mathbf{x})$. Thus

$$t^n \int f_N(\mathbf{x}) = \int f_N(t\mathbf{x}) \le L(\chi_{ta}) \le U(\chi_{tA}) \le \int g_N(t\mathbf{x}) = t^n \int g_N(\mathbf{x}).$$

Since A has volume,

$$\lim_{N\to\infty} t^n \int f_N = \lim_{N\to\infty} t^n \int g_N = t^n \operatorname{vol}_n A,$$

so, in particular, for any $\epsilon > 0$,

$$U(\chi_{tA}) - L(\chi_{tA}) < \epsilon,$$

so $U(\chi_{tA}) = L(\chi_{tA})$, so χ_{tA} is integrable, and

$$\operatorname{vol}_n(tA) = \int \chi_{tA} = t^n \operatorname{vol}_n(A). \quad \square$$

Page 412 In Exercise 4.1.5, part (d) and Exercise 4.1.6, part (c), a should be positive: $0 < a < b$.

Page 413 Exercise 4.1.14: Since the geometric mean for negative numbers is problematic, it would be better to define f as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \notin [0,1], \text{ or } x \text{ is rational} \\ 1 & \text{if } x \in [0,1], \text{ and } x \text{ is irrational.} \end{cases}$$

Page 416 The statement that an outcome with probability 0 will not occur may seem to contradict the statement, in the subsequent discussion of infinite, continuous sample spaces, that in such a setting "each individual outcome has probability 0." There is actually no contradiction. When a sample space is infinite, an individual outcome cannot occur because it is physically meaningless. We can think of spinning a bottle so that it ends up at exactly angle $\pi/2$, but we could never measure such a result. So, although it may seem obvious that each time we spin the bottle it lands on some angle, we really should think of it as landing within some measurable range of angles. It may seem peculiar that an infinite number of outcomes each with probably 0 can add up to something positive (in this case, 2π), but it is the same as the more familiar notion that a line has length, while the points that compose it have length 0.

Page 417 Margin note, third line from the bottom: "introducing them", not "introducing then".

Page 418 Second line after Equation 4.2.9: "the needle intersects," not "the needle intersect."

Line immediately above Equation 4.2.10: $\int_0^\pi \sin\theta\, |d\theta| = 2$. (It does not equal π.)

Page 419 Margin note, line 4: "to 20 feet," not "to20 feet."

Page 420 Line 8: there is an extra period after "data."

Page 421 Equation 4.2.15: this sums to 2, not 4/3! So in the next sentence, it should be "any sum smaller than \$2 . . . "

We get the result 2 as follows:

For $|x| < 1$, we have

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}, \quad \text{so} \quad \sum_{n=1}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2},$$

which gives

$$\sum_{n=1}^{\infty} n x^n = \frac{x}{(1-x)^2},$$

which gives

$$\sum_{n=1}^{\infty} n \cdot \frac{1}{2^n} = \frac{1/2}{(1/2)^2} = 2.$$

Page 422 Definition 4.2.12 of variance: There is an unfortunate typo in Equation 4.2.17; a $\mu(\mathbf{x})$ was omitted on the right-hand side. The equation should be

$$\mathrm{Var}\,(f) = E\Big(\big(f - E(f)\big)^2\Big) = \int_S \big(f(\mathbf{x}) - E(f)\big)^2 \mu(\mathbf{x}) |d^k\mathbf{x}|.$$

Page 424 In Equation 4.2.25, the $-t^2$ in the exponent should be $-x^2$:

$$\mu(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}.$$

Page 426 In the margin note about the error function, the 2π on the left should be $\sqrt{2\pi}$:

$$\frac{1}{\sqrt{2\pi}} \int_0^a e^{-\frac{t^2}{2}}\, dt = \frac{1}{2}\, \mathrm{erf}\left(\frac{a}{\sqrt{2}}\right).$$

Page 428 Exercise 4.2.5: The sample space is all of $\mathbb{R}$. Part (a) should have (x) at the end:

$$\mu(x) = \frac{1}{2a} \chi_{[-a,a]}(x)$$

Part (b) should be with the chapter review exercises, as it uses material from Section 4.11.

Page 429 Caption to Figure 4.3.2, last sentence: "The center region is black", not "The center region of is black".

Page 431 We are not consistent in our use of notation for graphs. In Definition 3.1.1 and on this page we use $\Gamma(f)$, but on page 433 we use Γ_f and on page 778 we use $\mathrm{gr}(f)$.

Page 436 First line of second paragraph: "in Definition measuredef" should be "in Definition 4.4.1."

Page 436 In Definition 4.4.1 (and in other definitions in the text), "if and only if" is not necessary. Mathematical definitions (unlike definitions in ordinary language) are always unambiguous. However, there are other ways to define measure 0; if one used a different definition, the statement of Definition 4.4.1 would still be true, but it would be a proposition, requiring proof, and the "if and only if" would be needed.

Page 437 In the last line of the proof of Theorem 4.4.3, the second equation should be

$$\sum_{i,j} \operatorname{vol} B_{i,j} \le \epsilon$$

(not $\operatorname{vol} X_1 \cup X_2 \cup \cdots \le \epsilon$).

Page 438 p.438 Line 10: Example 4.3.3, not 4.4.2:

... unlike the function of Example 4.3.3, which, as far as we know, is only a pathological example, devised to test the limits of mathematical statements.

Page 440 Second paragraph of the proof of Lemma 4.4.6: $|\mathbf{x}_j - \mathbf{y}_j|$, not $|f(\mathbf{x}_j) - f(\mathbf{y}_j)|$:

"Since $|\mathbf{x}_j - \mathbf{y}_j| \to 0$ as $j \to \infty$, the subsequence $\mathbf{y}_{j_k}$ also converges to $\mathbf{p}$."

The next paragraph would perhaps be clearer if the first sentence were:

"The function f is certainly not continuous at $\mathbf{p}$, so $\mathbf{p}$ has to be in a particular box, which we will call B_p."

Page 441 We should perhaps have reminded readers that $\exists$ means "there exist." The symbol was used in Section 0.2.

Page 454 Exercise 4.5.17, part (a): "Let $M_r(\mathbf{x})$ be the rth smallest ... ", not "Let $M_r(\mathbf{x})$ be the rth largest ... ".

Page 459 In Equation 4.6.14, the sum on the right should start at $i = 1$, not $i = -k$:

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{k} w_i\big(f(x_i) + f(-x_i)\big),$$

Page 465 Exercise 4.6.2: for $k = 1$, we meant the initial conditions to be $x_1 = .7$ and $x_2 = .5$ (not $x_1 = 17$ and $x_2 = .57$).

Page 467 Definition 4.7.2: We should have specified a *bounded* subset and a *finite* collection:

Definition 4.7.2 (A paving of $X \subset \mathbb{R}^n$). A paving of a bounded subset $X \subset \mathbb{R}^n$ is a finite collection $\mathcal{P}$ of subsets $P \subset X$ such that

$$\cup_{P \in \mathcal{P}} P = X, \text{ and } \operatorname{vol}_n(P_1 \cap P_2) = 0 \text{ (when } P_1, P_2 \in \mathcal{P} \text{ and } P_1 \neq P_2).$$

Page 468 In Definition 4.7.4 we use "diam" for "diameter," but we don't define it until page 487, just after Equation 4.9.9.

Page 468 At present, Theorem 4.7.5 requires f to be integrable. In a future edition, we will change Theorem 4.7.5 to something like

Theorem 4.7.5 (Integrals using arbitrary pavings) . *Let $X \subset \mathbb{R}^n$ be a bounded subset, and $\mathcal{P}_N$ be a nested partition of X. Suppose the boundary ∂X satisfies $\operatorname{vol}_n(\partial X) = 0$. Then $f : \mathbb{R}^n \to \mathbb{R}$ is integrable if and only if the upper and lower limits using the nested partition are equal:*

$$\lim_{N \to \infty} U_{\mathcal{P}_N}(f) = \lim_{N \to \infty} L_{\mathcal{P}_N}(f). \tag{4.7.4}$$

In that case, they are both equal to

$$\int_X f(\mathbf{x}) \, |d^n \mathbf{x}|. \tag{4.7.5}$$

This will solve some problems with the current proof of Theorem 4.9.1.

Page 472 Last margin note: Definition 2.1.11 does not exist. Column operations are defined by replacing the word "row" in Definition 2.1.1 of row operations by the word "column".

Page 475 The last margin note should be on page 476.

Page 478 Equation 4.8.36: Note that when we write this permutation as $(2, 3, 1)$ we are simply dropping the left-hand side, which carries no information.

Conflicting "shorthand" notation for permutations exist. As we describe it, the notation $(3, 1, 2)$ means that the first entry goes to third place, the second goes to first, and the third goes to second. But $(3, 1, 2)$ is often interpreted as the cyclical permutation "third goes to first, which goes to second, which goes back to third": $3 \to 1 \to 2 \to 3 \ldots.$

In this cyclical notation, the permutation $\begin{bmatrix}1\\2\\1\end{bmatrix} \mapsto \begin{bmatrix}3\\2\\1\end{bmatrix}$, which leaves the second entry unchanged, would be written $(3,1)$, i.e., $3 \to 1 \to 3$. The permutation that we would write $(3,5,1,4,2)$ would be written $(1,3)(2,5)$, or possible $(13)(25)$.

Page 478 Definition 4.8.15: We regret not having stated explicitly that $\operatorname{sgn}(\sigma \circ \tau) = \operatorname{sgn}\sigma \operatorname{sgn}\tau$:

$$\operatorname{sgn}(\sigma \circ \tau) = \det M_{\sigma\circ\tau} = \det(M_\sigma M_\tau) = \det M_\sigma \det M_\tau = \operatorname{sgn}\sigma \operatorname{sgn}\tau.$$

It was to get this equation easily that we defined the signature as we did, in terms of the determinant, which we had already defined in terms of its properties. The standard approach is to define the determinant in terms of the signature (turning Theorem 4.8.17 into a definition). This makes it excruciating to prove that $\operatorname{sgn}(\sigma\circ\tau) = \operatorname{sgn}\sigma \operatorname{sgn}\tau$, in order to get $\det A \det B = \det(AB)$. Of course, in mathematics, when you remove a difficulty in one place, it typically springs up someplace else; with our definition of the determinant, proving existence was not easy.

Page 480 In two places in the first line after Equation 4.8.44, $\operatorname{sgn}(\sigma)$ should be $\operatorname{sgn}(\sigma')$: "and the result follows from $\operatorname{sgn}(\tau^{-1} \circ \sigma') = \operatorname{sgn}(\tau^{-1})\big(\operatorname{sgn}(\sigma')\big) = -\operatorname{sgn}(\sigma')$, since ... "

Page 481 First line after Definition 4.8.19: one too many "is."

Page 484 Hint for Exercise 4.8.7: This hint is not actually used in the solution. Using the hint, one could write the following for part (a):

$$\det|\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{0}}, \dots, \vec{\mathbf{a}}_n| = \det|\vec{\mathbf{a}}_1, \dots, 2\vec{\mathbf{0}}, \dots, \vec{\mathbf{a}}_n| = 2\det|\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{0}}, \dots, \vec{\mathbf{a}}_n|,$$

which implies that the determinant must be 0.

Page 485 Last line of first paragraph: "volume of the parallelepiped," not area.

Page 488 In the equation following Equation 4.9.12, the left-hand side should be

$$U_{T(\mathcal{D}_N)}(\chi_{T(A)}) - L_{T(\mathcal{D}_N)}(\chi_{T(A)});$$

the upper and lower sums are with respect to the nested partition $T(\mathcal{D}_N)$.

Page 494 Discussion after Proposition 4.10.3: The reference should be to Corollary 4.3.10, not to Theorem 4.3.9. (That theorem concerns integrability, not the actual integral.)

Page 496 Last margin note: The sentence "At $\varphi = -\pi/2$ and $\varphi = \pi/2$, $r = 0$" should be deleted.

Page 502 Line 5: "in this case we can solve $xy = u$", not "in this case we can solve $y = u/v$".

Page 502 Bottom margin note: we mean to write Exercise 4.10.4, not 4.5.19.

Page 503 Exercise 4.10.3: This exercise should read

"Show that in complex notation, with $z = x + iy$, the equation of the lemniscate of Figure 4.10.3 can be written $|z^2 - \frac{1}{2}| = \frac{1}{2}$. Hint: See Example 4.10.19."

The equation given in the text is the equation for a different lemniscate.

Page 508 Caption: "first good fortune," not "first good fortunate."

Page 509 End of last margin note: "except on a set of measure 0", not "except on a measure 0."

The proof of Theorem 4.11.8 is not correct; the main idea is right but there is a fiddly problem with the truncations. Here is the rewritten proof:

Proof. Set $h_k = f_k - g_k$, and $H_l = \sum_{k=1}^{l} h_k$. The functions H_l form a sequence of Riemann-integrable functions converging to 0 except on a set of measure 0; if in addition they all have support in $B_R(\mathbf{0})$ and $|H_l| \le R$ for all l, then H_l meets the conditions for f_k in Theorem 4.11.4, so

$$\lim_{l\to\infty} \int_{\mathbb{R}^n} H_l(\mathbf{x})\,|d^n\mathbf{x}| = 0 \quad \text{i.e.,} \quad \lim_{l\to\infty} \sum_{k=1}^{l} \int_{\mathbb{R}^n} h_k(\mathbf{x})\,|d^n\mathbf{x}| = 0, \qquad 4.11.18$$

proving the result. We will reduce the general case, where H_l is not bounded with bounded support, to this one, by appropriately truncating the H_l.

Choose $\epsilon > 0$ and choose M such that

$$\sum_{k=M+1}^{\infty} \int_{\mathbb{R}^n} |h_k(\mathbf{x})||d^n\mathbf{x}| < \epsilon, \qquad 4.11.19$$

so that for $l > M$ we have

$$\int_{\mathbb{R}^n} |H_l(\mathbf{x}) - H_M(\mathbf{x})||d^n\mathbf{x}| \le \sum_{k=M+1}^{l} \int_{\mathbb{R}^n} |h_k(\mathbf{x})||d^n\mathbf{x}| \le \sum_{k=M+1}^{\infty} \int_{\mathbb{R}^n} |h_k(\mathbf{x})||d^n\mathbf{x}| < \epsilon.$$

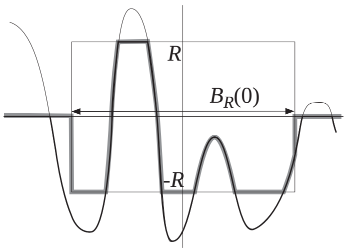

FIGURE 4.11.1B.

The thin line shows the graph of f, and the dark line shows $\inf(R\chi_{B_R(\mathbf{0})}, f)$. Next we take the sup of the dark line and $-R\chi_{B_R(\mathbf{0})}$, to get the thick, light gray line representing $[f]_R$.

Next choose R such that $\sup|H_M(\mathbf{x})| < R/2$ and $H_M(\mathbf{x}) = 0$ when $|\mathbf{x}| \geq R$. We will define the R-truncation of H_l by the formula

$$[H_l]_R = \sup\Big(-R\chi_{B_R(\mathbf{0})},\, \inf\big(R\chi_{B_R(\mathbf{0})}, H_l\big)\Big); \qquad 4.11.20$$

i.e., replace $H_l(\mathbf{x})$ by 0 if $|\mathbf{x}| > R$, by R if $H_l(\mathbf{x}) > R$, and by $-R$ if $H_l(\mathbf{x}) < -R$, as shown in Figure 4.11.1b. The $[H_l]_R$ form a sequence of Riemann-integrable functions all with support in $B_R(\mathbf{0})$ and all bounded by R, and tending to 0 except on a set of measure 0, so, by Theorem 4.11.4,

$$\underbrace{\lim_{l\to\infty}\int_{\mathbb{R}^n}[H_l]_R(\mathbf{x})\,|d^n\mathbf{x}| = 0.}_{\text{main motor of the proof}} \qquad 4.11.21$$

To prove Equation 4.11.17, we need to show that

$$\lim_{l\to\infty}\int_{\mathbb{R}^n} H_l(\mathbf{x})\,|d^n\mathbf{x}| = 0.$$

To do this, we consider H_l as the sum $[H_l]_R + H_l - [H_l]_R$ and consider separately the integral of $[H_l]_R$ (see Equation 4.11.21) and the integral of $H_l - [H_l]_R$ (the remainder of the proof).

At this point we have done most of the work (the hard part was proving Theorem 4.11.4). But, for $l > M$, we still need to deal with the difference $H_l - [H_l]_R = (H_l - H_M) - ([H_l]_R - H_M)$. We already know that the integral of $|H_l - H_M|$ is less than ϵ, so we only need to consider the integral of $|[H_l]_R - H_M|$. Outside $B_R(\mathbf{0})$ we have $H_M = 0$ and $[H_l]_R = 0$, so

$$\int_{\mathbb{R}^n - B_R(\mathbf{0})}\big|[H_l]_R - H_M(\mathbf{x})\big||d^n\mathbf{x}| = 0. \qquad 4.11.22$$

For the integral of $|[H_l]_R - H_M|$ inside $B_R(\mathbf{0})$, first find N such that $U_N(|H_l - H_M|) < \epsilon$. Then consider the union A of the cubes $C \in \mathcal{D}_N(\mathbb{R}^n)$ that intersect $B_R(\mathbf{0})$ and where $M_C(|H_l - H_M|) > R/2$. As shown in the margin, these have total volume at most $2\epsilon/R$. Let B be the union of the cubes $C \in \mathcal{D}_N(\mathbb{R}^n)$ that intersect $B_R(\mathbf{0})$ and such that $M_C(|H_l - H_M|) \leq R/2$; on these, $|H_l| \leq |H_l - H_M| + |H_M| \leq R/2 + R/2 = R$, so $[H_l]_R = H_l$. Thus

The upper sum $U_N(|H_l - H_M|)$ is small, so A must have small volume; in fact, it is at most $2\epsilon/R$:

$$\begin{aligned}\epsilon &> U_N|H_l - H_M|\\ &= \sum_{C\in\mathcal{D}_N(\mathbb{R}^n)} M_C|H_l - H_M|\operatorname{vol}_n C\\ &\geq \sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n)\\ C\subset A}} M_C|H_l - H_M|\operatorname{vol}_n C\\ &\geq \sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n)\\ C\subset A}} (R/2)\operatorname{vol}_n C\\ &= (R/2)\operatorname{vol}_n A.\end{aligned}$$

$$\begin{aligned}&\int_{B_R(\mathbf{0})}\Big|[H_l]_R(\mathbf{x}) - H_M(\mathbf{x})\Big||d^n\mathbf{x}| \qquad 4.11.23\\ &= \int_A\Big|[H_l]_R(\mathbf{x}) - H_M(\mathbf{x})\Big||d^n\mathbf{x}| + \int_B\Big|[H_l]_R(\mathbf{x}) - H_M(\mathbf{x})\Big||d^n\mathbf{x}|\\ &\leq \frac{3R}{2}\operatorname{vol}_n(A) + \int_B\Big|H_l(\mathbf{x}) - H_M(\mathbf{x})\Big||d^n\mathbf{x}| \leq 3\epsilon + \epsilon = 4\epsilon. \quad \square\end{aligned}$$

Page 510 Equation 4.11.19: We meant to write the sums with $k = m+1$, not $k = m$. (But it's correct as stated; the sums starting with $k = m$ are at least as big as the sums starting with $k = m+1$, so either way we can go from the third to the fourth lines of Equation 4.11.21.)

A somewhat more serious issue is that if $[f_k]_R = f_k$ and $[g_k]_r = g_k$, this does not imply $[f_k - g_k]_R = f_k - g_k$. The simplest way to fix this seems to be change

Equation 4.11.19, stating explicitly that we are choosing R big enough so that:

$$\sum_{k=1}^{m} f_k = \sum_{k=1}^{m}[f_k]_R, \qquad \sum_{k=1}^{m} g_k = \sum_{k=1}^{m}[g_k]_R, \qquad \sum_{k=1}^{m} f_k - g_k = \sum_{k=1}^{m}[f_k - g_k]_R. \tag{4.11.19}$$

The left side of Equation 4.11.22 should be an absolute value:

$$\left| \int_{\mathbb{R}^n} \sum_{k=1}^{p} [f_k - g_k]_{2R}(\mathbf{x}) \, |d^n\mathbf{x}| \right| < \epsilon. \tag{4.11.22}$$

We perhaps should have said that Equation 4.11.22 uses the dominated convergence theorem. We have

$$\lim_{p\to\infty} \int_{\mathbb{R}^n} \sum_{k=1}^{p}[f_k - g_k]_{2R}(\mathbf{x})|d^n\mathbf{x}| \underbrace{=}_{\text{dom. converg.}} \int_{\mathbb{R}^n} \underbrace{\Big(\lim_{p\to\infty} \sum_{k=1}^{p}[f_k - g_k]_{2R}(\mathbf{x})\Big)}_{\text{0 by hypothesis}} |d^n\mathbf{x}| = 0.$$

Page 511 First line after Equation 4.11.27: "at one point," not "at one points."

Page 514 The statement in the margin that "the union of sets of measure 0 has measure 0" is incorrect. It should be "the union of finitely many (or countably many) sets of measure 0 has measure 0."

Page 514 The proof of Proposition 4.11.14 is not correct. It should be as follows:

Proof. Suppose $f = \sum_{k=1}^{\infty} f_k$ and $g = \sum_{k=1}^{\infty} g_k$, with all f_k, g_k R-integrable, and

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(x)| \, dx < \infty, \quad \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |g_k(x)| \, dx < \infty. \tag{4.11.44}$$

Define $F_m = \sum_{k=1}^{m} f_k$ and $G_m = \sum_{k=1}^{m} g_k$, $H_m = \sup\{F_m, G_m\}$.

The problem is that the hypothesis $f \underset{L}{\leq} g$ does not imply that for m sufficiently large we have $F_m \underset{L}{\leq} G_m$: the inequality might go the other way on smaller and smaller sets. The solution will be to find new R-integrable functions h_k such that $g = \sum_{k=1}^{\infty} h_k$, and such that if we set $H_m = \sum_{k=1}^{m} h_k$, then indeed $F_m \underset{L}{\leq} H_m$ for all m.

Define $H_m = \sup\{F_m, G_m\}$, and $h_m = H_m - H_{m-1}$ (where we set $H_0 = 0$). finally $h_m = H_m - H_{m-1}$ (where we set $H_0 = 0$). Then certainly $F_m \leq H_m$, and

$$\sum_{m=1}^{\infty} h_m(\mathbf{x}) = \lim_{m\to\infty} H_m(\mathbf{x}) = g(\mathbf{x}). \qquad 4.11.45$$

Moreover,

$$|h_m(\mathbf{x})| = |H_m(\mathbf{x}) - H_{m-1}(\mathbf{x})| \le \sup\{|f_m(\mathbf{x})|, |g_m(\mathbf{x})|\} \le |f_m(\mathbf{x})| + |g_m(\mathbf{x})|.$$

We see the first inequality as follows. If the sup defining H is given by F (resp. G) for both m and $m-1$, clearly $h_m(\mathbf{x}) = f_m(\mathbf{x})$ (resp. $h_m(\mathbf{x}) = g_m(\mathbf{x})$). If it is given by F for m and by G for $m-1$, then

$$|h_m(\mathbf{x})| = |F_m(\mathbf{x}) - G_{m-1}(\mathbf{x})| \le |F_m(\mathbf{x}) - F_{m-1}(\mathbf{x})| = |f_m(\mathbf{x})| \qquad 4.11.46$$

and similarly in the fourth case. Thus

$$\sum_{m=1}^{\infty} \int_{\mathbb{R}^n} |h_m(\mathbf{x})||d^n\mathbf{x}| \le \sum_{m=1}^{\infty} \int_{\mathbb{R}^n} |f_m(\mathbf{x})||d^n\mathbf{x}| + \sum_{m=1}^{\infty} \int_{\mathbb{R}^n} |g_m(\mathbf{x})||d^n\mathbf{x}| < \infty. \qquad 4.11.47$$

Finally

$$\begin{aligned}\int_{\mathbb{R}^n} f(\mathbf{x})|d^n\mathbf{x}| &= \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\mathbf{x})|d^n\mathbf{x}| = \lim_{m\to\infty} \int_{\mathbb{R}^n} F_m(\mathbf{x})|d^n\mathbf{x}| \\ &\le \lim_{m\to\infty} \int_{\mathbb{R}^n} H_m(\mathbf{x})|d^n\mathbf{x}| = \int_{\mathbb{R}^n} g(\mathbf{x})|d^n\mathbf{x}|. \qquad \square\end{aligned}$$

Page 514 Proposition 4.11.15: We should have specified that a and b are constants.

Page 516 We should have mentioned that Theorems 4.11.19 and 4.11.20 are proved in Appendix A.21.

Page 518 In the last line in the margin, the third integral concerns f_2, not f_1:

$$\int f(x)dx = \int f_1(x)\,dx + i\int f_2(x)\,dx.$$

Page 520 Equation 4.11.73: An integral is missing on the second line. The equation should be

$$\begin{aligned}\lim_{h\to 0} \frac{\mathcal{L}f(s+h) - \mathcal{L}f(s)}{h} &= \lim_{h\to 0} \int_0^{\infty} \frac{e^{-(s+h)t} - e^{-st}}{h} f(t)\,dt \\ &= \lim_{h\to 0} \int_0^{\infty} f(t)e^{-st}\frac{e^{-ht}-1}{h}\,dt\end{aligned}$$

Page 521 Parts (b) of Exercises 4.11.6 and 4.11.7 are too difficult and should be deleted.

Page 522 Margin note: "not absolutely convergent," not "not absolutely convergence."

Page 526 Exercise 4.27: a sum was omitted from the definition of f. It should be

$$f(x) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{1}{\sqrt{|x - a_k|}}.$$