Errata and Clarifications: Chapter 3

Updated April 14, 2004

Page 294 Caption to Figure 3.1.3. We wrote that "the curve in $I_1 \times J_1$ can also be thought of as the graph of a function expressing $x \in I_1$ as a function of $y \in J_1$", but this is wrong, because we made J_1 too big; there are values of $y \in J_1$ that give no values in $I - 1$.

Page 297 Third line: "variables in terms," not "variables terms."

Page 300 There are several mistakes in margin notes. In the second line of the first note, $[\mathbf{D}F(\mathbf{a})]$should be $[\mathbf{DF}(\mathbf{a})]$. The next note has the reverse problem: the $\mathbf{F}$ should be F; in the case of a curve in the plane, $\mathbb{R}^{n-k} = \mathbb{R}$ and $\mathbf{F}$ is the single function F. The last note is wrong; it should read, "More generally, for an $(n-1)$-dimensional manifold in any $\mathbb{R}^n$, … "

Page 305 Line 5 should read: " … good picture of a parametrized curve or surface" not " … good picture of parametrized the curve or surface."

Page 307 (comment, not correction) Example 3.1.17: In this interpretation, $\gamma'(t)$ is the *velocity vector*; it is tangent to the curve at $\gamma(t)$ and its length is the speed at which you are traveling at time t.

Five lines from the bottom, "The requirement that $[\mathbf{D}\gamma(\mathbf{u})]$ be one" should be "The requirement that $[\mathbf{D}\gamma(\mathbf{u})]$ be one to one," and

$$\vec{\gamma}'(t) \neq \mathbf{0}'' \quad \text{should be} \quad \vec{\gamma}'(t) \neq \mathbf{0}.$$

Page 308 Second line of the remark, "it may look as though", not "it may looks as though".

Page 315 Exercise 3.1.24, second line: "a smooth curve", not "is a smooth curve".

Exercise 3.1.26: we used A both to denote the $A(n, n)$ (the space of antisymmetric $n \times n$ matrices) and to denote the matrice A. The matrix A is $n \times n$.

Page 316 Exercise 3.1.28, part (c): $\mathbf{g}$, not g.

Definition 3.2.1 is not stated correctly. It should be

Definition 3.2.1 (Tangent space of a manifold). Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold, so that near $\mathbf{z} \in M$, M is the graph of a C^1 mapping $\mathbf{f}$ expressing $n-k$ variables as functions of the other k variables. If $\mathbf{z} = \mathbf{a}+\mathbf{f}(\mathbf{a})$, then the tangent space to M at $\mathbf{z}$, denoted $T_{\mathbf{z}}M$, is the graph of $[\mathbf{Df}(\mathbf{a})]$.

What does $\mathbf{z} = \mathbf{a} + \mathbf{f}(\mathbf{a})$ mean? The point $\mathbf{a}$ is in a k-dimensional subset of $\mathbb{R}^n$; it has n entries but $n-k$ of them are 0. Similarly, $\mathbf{f}(\mathbf{a})$ has n entries but k of them are 0. So we are adding two n-dimensional points to get a third n-dimensional point. What makes us slightly uneasy is that we aren't supposed to add points, just vectors. We would prefer to write $\mathbf{z} = \begin{pmatrix} \mathbf{a} \\ \mathbf{f}(\mathbf{a}) \end{pmatrix}$ but that would be assuming that the k "active" variables comes first, which isn't necessarily the case.

Page 317 The title to Example 3.2.2 should be "Tangent line and tangent space to smooth curve", not "Tangent line and tangent plane ... ".

Page 318 Example 4.2.3, second line of second paragraph: "playing the role of $\mathbf{x}$" (not $\mathbf{x}_1$).

Page 319 Example 3.2.5 refers to Example 3.1.11, but that example concerned a different function. Example 3.2.5 has been rewritten to show that X_c is a smooth curve for all c:

The locus X_c defined by $x^9 + 2x^3 + y + y^5 = c$ is a smooth curve for all values of c since the derivative of the function $F\begin{pmatrix} x \\ y \end{pmatrix} = x^9 + 2x^3 + y + y^5$ is

$$\left[\mathbf{D}F\begin{pmatrix} x \\ y \end{pmatrix}\right] = [9x^8 + 6x^2\,,\, 1 + 5y^4],$$

and $1 + 5y^4$ is never 0.

Page 320 We should have said that $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Page 321 Equation 3.2.19 should end with $= \mathbf{0}$:

$$[D_{j_1}\mathbf{F}(\mathbf{c}), \dots, D_{j_{n-k}}\mathbf{F}(\mathbf{c})]\dot{\mathbf{x}} + [D_{i_1}\mathbf{F}(\mathbf{c}), \dots, D_{i_k}\mathbf{F}(\mathbf{c})]\dot{\mathbf{y}} = \mathbf{0}.$$

Page 322 Exercise 3.2.5: parts (a) and (b) not (a) and (c)

Page 325 In the first line after Equation 3.3.10, the reference should be to Equation 3.3.9 and footnote 7.

Page 326 First line after Equation 3.3.16: "There are 30 such terms" refers to terms *other* than the five terms in Equation 3.3.16. Thus there are 35 in all.

Page 332 The first term in the 4th line should have a minus sign:

$$-4y\sin(x + y^2).$$

Page 332 Line immediately before Equation 3.3.38: "$(-\frac{1}{3}!)h_1^3$" should be "$(-\frac{1}{3!})h_1^3$."

Page 334 Exercise 3.3.6: in part (b), the hypothesis $f(-\mathbf{x}) = -f(\mathbf{x})$ should have been included.

Page 335 Exercise 3.3.14: It's possible to solve this using partial derivatives (and a computer), but it's much easier with the techniques of Section 3.4; the exercise should be with the exercises for that section.

In part (b), $D^{[1,1,1]}$ should be $D_{[1,1,1]}$.

Page 336 In the main text, Edmund Landau's dates are given incorrectly. They are correct in the margin note.

Lines 2 and 3 from bottom: "We will write them only near 0, but by translation they can be written at any point where the function is defined" (not " . . . they can be written anywhere").

Page 337 The first margin note suggests, incorrectly, that all odd functions and all even functions have Taylor polynomials. It should read

"The Taylor function of an odd function can have only odd terms, and the Taylor function of an even function can have only even terms."

Page 340 Equation 3.4.17 should include $= 0$:

$$F\begin{pmatrix} x \\ y \end{pmatrix} = x^3 + xy + y^3 - 3 = 0.$$

Page 342 Exercise 3.4.4 should read

Find numbers a, b, c such that when f is C^3,

$$h\Big(af(0) + bf(h) + cf(2h)\Big) - \int_0^h f(t)\,dt \in o(h^3).$$

(If you omit the factor h, then a, b, c are not numbers, but multiples of h.)

Exercise 3.4.5 should read

Find numbers a, b, c such that when f is C^3,

$$h\Big(af(0) + bf(h) + cf(2h)\Big) - \int_0^{2h} f(t)\,dt \in o(h^3).$$

Page 343 In the third line, the equality sign should be raised: "But $p(\mathbf{x}) = x_1x_2x_3$" (not But $p(\mathbf{x})_{=}x_1x_2x_3$").

In the displayed equation in the margin, $Q(t)$ should be $Q(f)$:

$$Q(f) = \int_0^1 (f(x))^2\,dx.$$

The bottom margin note is incorrect; the theorem described is due to Fermat but it is not Fermat's little theorem.

Page 345 Equation 3.5.6 should have a "plus or minus":

$$\sqrt{a}x + \frac{b}{2\sqrt{a}} = \pm\sqrt{\frac{b^2 - 4ac}{4a}}.$$

Page 347 The margin note halfway down the page should specify a quadratic form on $\mathbb{R}^n$:

"Definition 3.5.9 is equivalent to saying that a quadratic form on $\mathbb{R}^n$ is positive definite if its signature is $(n, 0)$ and negative definite if its signature is $(0, n)$."

A quadratic form on $\mathbb{R}^n$ with signature $(k, 0)$, $k < n$, is not positive definite.

Margin note beginning "Definition 3.5.9 is equivalent": Exercise 3.5.7 concerns only positive definite quadratic forms.

The last margin note, about $Q(p)$, should be on page 343.

Pages 348–349 In several places – Equations 3.5.23, 3.5.25, and 3.5.28, and in the sentence before Equation 3.5.25 – we write things of the form $\alpha_1(\mathbf{x})^2$ which would be better written with an additional set of parentheses: $(\alpha_1(\mathbf{x}))^2$.

Page 351 Exercise 3.5.1: "finally the terms in x," not " ... in y."

Page 356 Last margin note: This is true for a quadratic form on $\mathbb{R}^n$.

Page 357 First margin note: This is true for a quadratic form on $\mathbb{R}^n$.

Page 360 Exercise 3.6.5, part (a) should read

(a) Find the critical points of the function $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + yz - xz + xyz.$

Page 362 There should be a $\triangle$ to mark the end of Example 3.7.3.

Page 363 Caption to Figure 3.7.2: There should be no comma after $-\mathbf{a} = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$.

Page 365 The $\triangle$ at the end of the caption should be on the next page, at the end of the example.

Page 366 Third paragraph of Example 3.7.6: The reference should be to Definition 3.1.16, not 3.1.18.

Page 367 Margin note immediately after the figure caption: "manifold," not "manifolds."

Page 368 Example 3.7.9 contains various errors. It should read as follows:

Example 3.7.9 (Critical points of functions constrained to ellipse). Let us follow this procedure for the function $f(\mathbf{x}) = x^2 + y^2 + z^2$ of Example 3.7.4, constrained as before to the ellipse given by

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - 1 \\ x - z \end{pmatrix} = \mathbf{0}. \qquad 3.7.19$$

We have

$$[Df(a)] = [2x, 2y, 2z],\ [DF_1(a)] = [2x, 2y, 0],\ [DF_2(a)] = [1, 0, -1],$$

so Theorem 3.7.7 says

$$[2x, 2y, 2z] = \lambda_1[2x, 2y, 0] + \lambda_2[1, 0, -1], \qquad 3.7.20$$

which gives

$$2x = \lambda_1 2x + \lambda_2, \quad 2y = \lambda_1 2y + 0, \quad 2z = \lambda_1 \cdot 0 - \lambda_2. \qquad 3.7.21$$

If $y \neq 0$, this gives $\lambda_1 = 1$, $\lambda_2 = 0$, and $z = 0$. The equations $F_1 = 0$ and $F_2 = 0$ then say that $x = 0$, $y = \pm 1$, so $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ are critical points. But if $y = 0$, then $F_1 = 0$ and $F_2 = 0$ give $x = z = \pm 1$. So $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$, with $\lambda_1 = 2$ and $\lambda_2 = \mp 2$, are also critical points.

Since our constraint is a compact manifold, the maximum and minimum values of f restricted to $\mathbf{F} = \mathbf{0}$ are attained at constrained critical points of f. Since $f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = f\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = 1$ and $f\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = f\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = 2$, we see that f achieves its maximum value of 2 at $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and at $\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and its minimum value of 1 at $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$. $\triangle$

Page 370 There should be a $\triangle$ to mark the end of Example 3.7.11.

Page 372 First paragraph: "the principal axis theorem," not "the principle axis theorem."

Page 374 Theorem 3.7.16 should read

Theorem 3.7.16. *A quadratic form Q_A has signature (k, l), if and only if A has k linearly independent eigenvectors with positive eigenvalues and l linearly independent eigenvectors with negative eigenvalues.*

The last margin note should refer to Equation 3.7.55 (not 3.7.54) and should be on the next page.

Page 375 Last margin note: The hint is for both parts of Exercise 3.7.7, not just part (b).

Page 376 Exercise 3.7.8: For $a, b \geq 0$.

Exercise 3.7.11: the parts are mislabeled: (d) should be (c), etc.

Exercise 3.7.13: "closest to and furthest from", not "closest and furthest from".

Exercise 3.7.14: We should have said "the unit circle".

Page 378 Equation 3.8.5: $g(X)$, not $g(x)$.

Page 379 Example 3.8.3, next to last line: the curvature $\frac{2}{5\sqrt{5}}$ is about 0.179, not 0.896. (We had put the $\sqrt{5}$ in the numerator.) We have redone Figure 3.8.2 below.

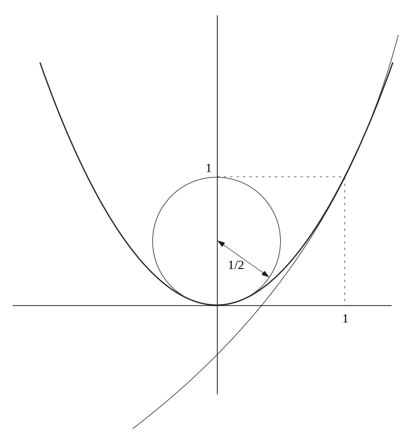

FIGURE 3.8.2. At $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, which corresponds to $a = 1$, the parabola given by $y = x^2$ looks much flatter than the unit circle. Instead, it resembles a circle of radius $5\sqrt{5}/2 \approx 5.59$. (A portion of such a circle is shown. Note that it crosses the parabola. This is the usual case, occurring when, in adapted coordinates, the cubic terms of the Taylor polynomial of the difference between the circle and the parabola are nonzero.) At the origin, which corresponds to $a = 0$, it has curvature 2 and resembles the circle of radius 1/2, which also has curvature 2. "Resembles" is an understatement. At the origin, the Taylor polynomial of the difference between the circle and the parabola starts with fourth-degree terms.

Page 384 Caption for Figure 3.8.6: The discussion of the second and third goats should read

"The second goat is thin. He lives on the top of a hill, with positive Gaussian curvature; he can reach less grass. The third goat is fat. His surface has negative Gaussian curvature; with the same length chain, he can get at more grass. This would be true even if the chain were so heavy that it lay on the ground."

Page 385 Proposition 3.8.9 doesn't apply to a surface known in "best" coordinates, where the Taylor polynomial starts with quadratic terms; in that case the linear terms a_1 and a_2 would be 0.

Page 387 Equation 3.8.42: The numerator should be $H(a^2 - b^2)$, not $H(b^2 - a^2)$.

Page 392 The fourth line of Equation 3.8.68 should be

$$= \Big(-\big(\kappa\big(s(t)\big)\big)^2 \big(s'(t)\big)^3 + s'''(t) \Big) \vec{\mathbf{t}}(s(t))$$

For consistency, the last line of Equation 3.8.68 should be $\vec{\mathbf{b}}\big(s(t)\big)$, not $\vec{\mathbf{b}}$.

Page 393 Exercise 3.8.3: show that the absolute value of the mean curvature of the unit sphere is 1 and that the Gaussian curvature is 1.

Page 394 In the hint for Exercise 3.8.11, we neglected to define $SO(3)$. It is the space of orthogonal 3×3 matrices with determinant $+1$. (Recall that an orthogonal $n \times n$ matrix is a matrix whose columns form an orthonormal basis of $\mathbb{R}^n$.)

Page 398 Exercise 3.21, part (a): $2d \cos \varphi$ should be $2ad \cos \varphi$:

$$a^2 + d^2 - 2ad \cos \varphi \; = \; b^2 + c^2 - 2bc \cos \psi.$$