# Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach

Errata and Clarifications for the Second Edition

## Chapter 2

## updated March 8, 2005

**Page 171** Margin note next to Equation 2.1.4: $[A\vec{\mathbf{b}}]$ should be $[A\,|\,\vec{\mathbf{b}}]$. Bottom margin note: to be consistent with later notation, we should write $[A\,|\,\vec{\mathbf{b}}]$, not $[A\,,\,\vec{\mathbf{b}}]$ and $[A'\,|\,\vec{\mathbf{b}}']$, not $[A', \vec{\mathbf{b}}']$.

**Page 172** Theorem 2.1.3: To be consistent with later notation, we should write $[A\,|\,\vec{\mathbf{b}}]$, not $[A\,,\,\vec{\mathbf{b}}]$ and $[A'\,|\,\vec{\mathbf{b}}']$, not $[A', \vec{\mathbf{b}}']$.

First margin note: $[A\,|\,\tilde{\mathbf{b}}]$, not $[A\tilde{\mathbf{b}}]$

We use the vertical line to avoid confusion with the *product* $A\vec{\mathbf{b}}$. You should not think that $\vec{\mathbf{b}}$ is somehow special as far as row reduction is concerned; the rules of row reduction apply equally to all the columns of $[A\,|\,\vec{\mathbf{b}}]$: the columns of $A$ and the column $\vec{\mathbf{b}}$.

**Page 176** Exercise 2.1.5, "in the algorithm for row reduction" should be "in Definition 2.1.1 of row operations".

**Page 178** Margin note: The vector $\vec{\mathbf{b}}$ does not contain the solutions.

**Pages 178, 179, 181** As for Page 172, to keep notation consistent, various augmented matrices should have vertical lines, not commas, as in $[\widetilde{A}\,|\,\widetilde{\vec{\mathbf{b}}}]$.

**Page 179** In the second line of Theorem 2.2.4, the $\mathbf{x}$ should be $\vec{\mathbf{x}}$: $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, not $A\mathbf{x} = \vec{\mathbf{b}}$.

In the remark, we mention linear independence prematurely; it is not discussed until Section 2.4.

**Page 181** In the proof of Theorem 2.2.4, the $\mathbf{x}$ should be $\vec{\mathbf{x}}$: $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, not $A\mathbf{x} = \vec{\mathbf{b}}$.

**Page 185** Exercise 2.2.6, part (a) should read: "For what values of $a$ does the system of equations in the margin have a solution?" (not "have a unique solution").

**Page 188** Part (3) of Definition 2.3.6: "$i \neq j$", not $1 \neq j$.

**Page 195** Definition 2.4.5 was perhaps not clear; we mean *an* vector $\vec{\mathbf{w}}$, not some particular $\vec{\mathbf{w}}$. Here is a rewrite:

**Definition 2.4.5 (Linear independence).** The vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k \in \mathbb{R}^n$ are linearly independent if every vector in $\mathbb{R}^n$ can be written as a linear combination of $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ in at most one way, i.e.:

$$\sum_{i=1}^{k} x_i \vec{\mathbf{v}}_i = \sum_{i=1}^{k} y_i \vec{\mathbf{v}}_i \quad \text{implies} \quad x_1 = y_1,\ x_2 = y_2, \dots, x_k = y_k.$$

**Page 196** First margin note, 4th line after the matrices: "is upper triangular with nonzero entries on the diagonal", not "is upper triangular form with nonzero entries on the diagonal".

**Page 205** Exercise 2.4.11 should be with the exercises for Section 2.5.

**Page 206** Part (a) of Exercise 2.4.13 was poorly stated. It should be:

(a) For $n = 1, n = 2, n = 3$, write the system of linear equations which the $a_{0,n}, \dots, a_{n,n}$ must satisfy so that the integral of 1 is exact, the integral of $x$ is exact, and so on, until you get to $x^n$.

Exercise 2.5.14, part (c): $W$ should be $W_t$.

**Page 212** Corollary 2.5.11: Rather than "i.e., if the kernel is zero" it would be better to say, "i.e., if the kernel has dimension 0."

**Page 214** In the second box (giving equivalent statements about a one-to-one linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$), statement 6 is incorrect. It should be:

The row-reduced matrix $\widetilde{A}$ has no nonpivotal column.

**Page 222** The parts of Exercise 2.5.7 are listed as (a), (b), (c), (b). Of course the second (b) should be (d).

**Page 224** Part (c) of Exercise 2.5.20: "For any vectors $\vec{\mathbf{b}} \in \mathbb{R}^n$", not "for any numbers $\vec{\mathbf{b}} \in \mathbb{R}^n$".

There is also an extra period in the margin note.

**Page 226** Fourth line of Example 2.6.3: a space is needed between "Example 2.6.2" and "and."

**Page 231** A plus sign is missing from Equation 2.6.18. It should be

$$\mathbf{v}'_i = p_{1,i}\mathbf{v}_1 + p_{2,i}\mathbf{v}_2 + \cdots + p_{n,i}\mathbf{v}_n.$$

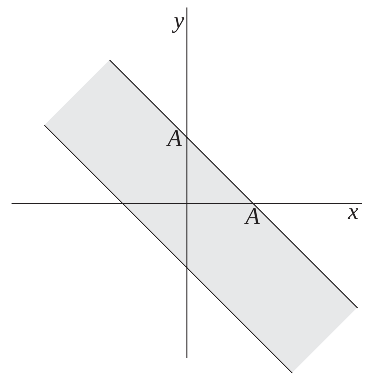
**Page 236** Exercise 2.6.3: The four matrices do not form a basis, since $\underline{\mathbf{v}}_3 = -\underline{\mathbf{v}}_4$.

Exercise 2.6.5: After the displayed equation, $\Phi^{-1}_{\{\underline{\mathbf{v}}\}}$ should be $\Phi_{\{\underline{\mathbf{v}}\}}$: "so that $\Phi_{\{\underline{\mathbf{v}}\}}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$."

**Page 237** Exercise 2.6.11: $A_a$ should be $A$.

**Page 243** In Definition 2.7.4 we should have required that $U \subset \mathbb{R}^n$ be open.

**Page 244** The way Figure 2.7.5 is drawn, the shaded strip seems to end in quadrants 2 and 4; actually, it is infinite. The following may suggest reality better:




**Page 246** In Example 2.7.11, we use a different order for the subscripts of $c$ than that given in Proposition 2.7.10. To make the text consistent, $c_{2,2,1}$ should be $c_{1,2,2}$ and $c_{1,1,2}$ should be $c_{2,1,1}$:

$$|D_2D_2\mathbf{f}_1| \le 3A = \underbrace{c_{1,2,2}}_{\substack{\text{bound for}\\ |D_2D_2\mathbf{f}_1|}} \quad \text{and} \quad |D_1D_1\mathbf{f}_2| \le 3A = \underbrace{c_{2,1,1}}_{\substack{\text{bound for}\\ |D_1D_1\mathbf{f}_2|}}$$

with all others 0, so

$$\sqrt{c^2_{1,2,2} + c^2_{2,1,1}} = 3A\sqrt{2}. \qquad 2.7.40$$

**Page 246** Four lines from the bottom– one reader wondered whether "blunderbuss" was "a new word from generation X". Our dictionary defines a blunderbuss as an "old-fashioned, short gun with large bore and flaring mouth, used for scattering shot at close range". It will hit a big target, but is not precise.

**Page 249** Statement of Theorem 2.7.13: in the next-to-last line, it should be "has a unique solution in the closed ball $\overline{U_0}$". To see why this is necessary, consider Example 2.8.1, where Newton's method converges to 1, which is not in $U_0$ but is in its closure.

**Page 249** In the bottom margin note, we discuss the importance of making sure both sides of an equation have the same units. In chemical engineering, fluid mechanics, etc., this is called "dimensional analysis."

**Page 250** At the end of Equation 2.7.55 we should write $< 1.2$, not $< 2$:

$$\left|[\mathbf{D}\vec{F}(\mathbf{a}_0)]^{-1}\right|^2 = \frac{1}{(\cos 2 - 1)^2}\Big((\cos 2)^2 + 1 + (1 - \cos 2)^2\Big) \sim 1.1727 < 1.2, \quad 2.7.55$$

Equation 2.7.56 contains several errors. In the second rows of the matrices on the right, two minus signs should be pluses. In the third line of Equation 2.7.56, the first $=$ in the last line should be $\leq$, the 4 under the square root should be 8, and the 2 after the second $=$ should be $2\sqrt{2}$.

The equation should be:

$$\begin{aligned}\left|\left[\mathbf{D}\vec{F}\begin{pmatrix}x_1\\y_1\end{pmatrix}\right] - \left[\mathbf{D}\vec{F}\begin{pmatrix}x_2\\y_2\end{pmatrix}\right]\right| &= \left|\begin{bmatrix}-\sin(x_1-y_1)+\sin(x_2-y_2) & \sin(x_1-y_1)-\sin(x_2-y_2)\\ \cos(x_1+y_1)-\cos(x_2+y_2) & \cos(x_1+y_1)-\cos(x_2+y_2)\end{bmatrix}\right|\\ &\leq \left|\begin{bmatrix}|-(x_1-y_1)+(x_2-y_2)| & |(x_1-y_1)-(x_2-y_2)|\\ |(x_1+y_1)-(x_2+y_2)| & |(x_1+y_1)-(x_2+y_2)|\end{bmatrix}\right|\\ &\leq \sqrt{8\big((x_1-x_2)^2+(y_1-y_2)^2\big)} = 2\sqrt{2}\left|\begin{pmatrix}x_1\\y_1\end{pmatrix} - \begin{pmatrix}x_2\\y_2\end{pmatrix}\right|. \qquad 2.7.56\end{aligned}$$

(Going from the second to the third line of Equation 2.7.56 uses $(a+b)^2 \leq 2(a^2+b^2)$.)

**Page 251** In the first line, $M = 2\sqrt{2}$, not $M = 2$.

Equation 2.7.57 should be:

$$|\vec{F}(\mathbf{a}_0)|\left|[\mathbf{D}\vec{F}(\mathbf{a}_0)]^{-1}\right|^2 M \leq .1 \cdot 1.2 \cdot 2\sqrt{2} \approx .34 < .5. \qquad 2.7.57$$

**Page 253** As on page 246, the order of subscripts for $c$ is wrong in three places at the bottom of the page. Below, the starred entries have been corrected:

$$\begin{array}{ll}\sup|D_1D_1f_1| \leq 3 = c_{1,1,1} & *\sup|D_1D_1f_2| = 0 = c_{2,1,1}\\ \sup|D_1D_2f_1| \leq 1 = c_{1,2,1} & *\sup|D_1D_2f_2| = 0 = c_{2,2,1}\\ *\sup|D_2D_2f_1| \leq 1 = c_{1,2,2} & \sup|D_2D_2f_2| = 2 = c_{2,2,2}.\end{array}$$

**Page 255** Exercise 2.7.3 involves showing that a function is Lipschitz, but we did not actually define a Lipschitz function in the text. If $X \subset \mathbb{R}^n$, then a mapping $\mathbf{f} : X \to \mathbb{R}^m$ is Lipschitz if there exists $C$ such that

$$\big|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\big| \leq C|\mathbf{x} - \mathbf{y}|.$$

(Of course a Lipschitz mapping is continuous; it is better than continuous.)

**Page 256** The margin note about Exercise 2.23 belongs on page 288.

Exercise 2.7.11 is missing part (b):

(b) Prove that this Newton's method converges.

**Page 260** There should be no vector $\vec{\mathbf{v}}$ in Definition 2.8.6. The definition should read

**Definition 2.8.6 (The norm of a linear transformation).** Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The norm $\|A\|$ of $A$ is

$$\|A\| = \sup |A\vec{\mathbf{x}}|, \text{ when } \mathbf{x} \in \mathbb{R}^n \text{ and } |\vec{\mathbf{x}}| = 1. \qquad 2.8.11$$

**Page 261** On the second line of Example 2.8.9 we say that the norm is $\frac{1+\sqrt{5}}{2}$; in Equation 2.8.8 we compute the norm as $\sqrt{\frac{3+\sqrt{5}}{2}}$. Both, of course, are correct, since

$$\sqrt{\frac{3+\sqrt{5}}{2}} = \sqrt{\frac{6+2\sqrt{5}}{4}} = \frac{1+\sqrt{5}}{2}.$$

**Page 264** Execise 2.8.8: In the displayed equation, $D$ should be $D^2$:

$$\|A\| = \left( \frac{|A|^2 + \sqrt{|A|^4 - 4D^2}}{2} \right)^{1/2}.$$

Fourth line from bottom: "mainly" should be "namely".

**Page 265** First sentence after Theorem 2.9.2: Exercise A.7.1, not 7.1.

**Page 270** We never proved Equation 2.9.13! Moreover, it is wrong, which shows how dangerous it is to omit proofs. The correct equation is

$$R_1 \;=\; R|L^{-1}|^2 \left( \sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}} - |L| \right).$$

**Proof.** Suppose $|\mathbf{x} - \mathbf{x}_0| < R_1$. Then

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| \;\leq\; |\mathbf{x} - \mathbf{x}_0| \sup |[\mathbf{Df}(\mathbf{x})]| \;\leq\; R_1 \sup |[\mathbf{Df}(\mathbf{x})]|.$$

We find a bound for $|[\mathbf{Df}(\mathbf{x})]|$:

$$|[\mathbf{Df}(\mathbf{x})] - [\mathbf{Df}(\mathbf{x}_0)]| = |[\mathbf{Df}(\mathbf{x})] - L| \underbrace{\leq}_{\text{Eq. 2.9.11}} \frac{1}{2R|L^{-1}|^2} |\mathbf{x} - \mathbf{x}_0| \leq \frac{R_1}{2R|L^{-1}|^2}$$

so

$$|[\mathbf{Df}(\mathbf{x})]| \leq |L| + \frac{R_1}{2R|L^{-1}|^2}, \quad \text{i.e.,} \quad \sup |[\mathbf{Df}(\mathbf{x})]| = |L| + \frac{R_1}{2R|L^{-1}|^2}.$$

Therefore (remember that $R$ is the radius of $V$, the domain of $\mathbf{g}$) we want to find the largest $R_1$ satisfying

$$R \;\geq\; \left(|L| + \frac{R_1}{2R|L^{-1}|^2}\right) R_1.$$

The right-hand side is 0 when $R_1 = 0$ and then increases as $R_1$ increases, so we want the largest value of $R_1$ for which the inequality is an equality. Thus we want to solve the quadratic equation

$$R_1^2 + 2R|L^{-1}|^2\,|L|R_1 - 2R^2|L^{-1}|^2 = 0,$$

which gives

$$R_1 = R|L^{-1}|^2\left(-|L| + \sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}}\right).$$

**Page 271** At the end of the first paragraph after Figure 2.9.6: "look at condition (3a) of the theorem" should be "look at condition (1) of the theorem." In the next paragraph, "condition (3b) is more delicate" should be "condition (2) is more delicate."

**Page 271** Last line: "an inverse function," not "a inverse function."

**Page 273** In the first line after Equation 2.9.18, the reference is to the wrong equation. "Next we need to compute the Lipschitz ratio $M$ (Equation 2.9.24)" should be "Next we need ... (Equation 2.9.11)."

**Pages 277, 278** The margin note about Equation 2.9.30 (page 277) should be on page 278.

**Page 280** The second margin note is completely false; we have no idea what we were thinking of. Using the second partial derivative method in Example 2.9.15 is perfectly possible and gives a Lipschitz ratio of $2\sqrt{3}$.

**Page 281** Second line: "diagonal matrices" should be "diagonal entries."

**Page 284** In Execise 2.9.4, the matrix $\begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$ should be $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$. This matrix appears three times.

**Page 286** The parts of Exercise 2.11 are mislabeled. They should be (a), (b), (c).

**Page 288** The margin note on page 256 about Exercise 2.23 belongs on this page.

**Page 290** Exercise 2.33: in two places, "of degree" should be "of degree at most":

"$q_1$ and $q_2$ are polynomials of degrees at most $k_2 - 1$ and $k_1 - 1$"

and

"the space of polynomials of degree at most $k_1 + k_2 - 1$."

(Say we have a polynomial $ax^2 + bx + c$. For it to live in a vector space, we have to allow for the possibility that $a = 0$. But then it is a first degree polynomial.) For the second sentence of the exercise, where we discuss $p_1$ and $p_2$, we don't have to say "at most" because those are specific polynomials. But $q_1$ and $q_2$ are variables.