Vector Calculus, Linear Algebra, and Differential Forms: A Unified Approach

Errata and Clarifications for the Second Edition

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Inside front cover

inf: infimum, not minimum

Preface

Page xv 16th line from bottom: "some of the material," not "some of material."

Chapter 0

Page 4 Second and third lines from the bottom: To be punctilious, we should have said "for all $y \in \mathbb{R}$:

"For all $x \in \mathbb{R}$ and for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $y \in \mathbb{R}$, if $|y - x| < \delta$, then $|y^2 - x^2| < \epsilon$."

Margin note about Bowditch: "American Bowditch," not "America ... "

Page 5 There are some inconsistencies of notation. In future editions we will write Equation 0.2.2 as

$$\text{The opposite of} \quad (\forall\, x) P(x) \quad \text{is} \quad (\exists x) \text{ not } P(x).$$

(But if we had something more complicated than $P(x)$ we would put it in parentheses.)

Most mathematicians avoid the symbolic notation, instead writing out quantifiers in full. But when there is a complicated string of quantifiers, they often use the symbolic notation to avoid ambiguity.

Page 7 It follows from the definition of $\subset$ that the empty set $\emptyset$ is a subset of every set.

We should have included an eighth "word":

$=$ "equality"; $A = B$ if A and B have the same elements

and specified that the symbol $\notin$ ("not in") means "not an element of"; similarly, $\not\subset$ means "not a subset of" and $\neq$ means "not equal".

We should also have noted that the order in which elements of a set are listed (even assuming they are listed) does not matter, and that duplicating does not affect the set; for example, $\{1, 2, 3\} = \{1, 2, 3, 3\}$.

Page 12 Example 0.4.4: The first sentence after the displayed equation should be

"This can be evaluated only if $x^2 - 3x + 2 \geq 0$, which happens if $x \leq 1$ or $x \geq 2$."

($x \leq 1$, not $x \leq 2$; $x \geq 2$, not $x \geq 3$.)

The second sentence should be

"So the natural domain is $(-\infty, 1] \cup [2, \infty)$."

Page 15 Example 0.4.10: The range of f should be the real numbers, not the real positive numbers. (The *image* of f is of course the real positive numbers.) Speaking of $f^{-1}(A)$ does not require, or even suggest, that A is a subset of the image of f. As a rule, finding the image of f is difficult, and it would drastically restrict the language of $f^{-1}(A)$ to make such a requirement.

Page 18 First line: "in this section" should be "in this section and in Appendix A.1."

Definition 0.5.1: The least upper bound is also known as the supremum. Definition 0.5.2: The great lower bound is also known as the infimum.

Page 18 The definition of the truncation $[x]_k$ and the discussion in the following paragraph of one number being larger than another fails to take into account the non-fractional part. Here is the rewritten version:

We denote by $[x]_k$ the number formed by keeping all digits to the left of and including the 10^{-k} column and setting all others to 0. Thus if $x = 5\,129.359\ldots$, then $[x]_{-2} = 5\,100.00\ldots$, $[x]_{-1} = 5\,120.00\ldots$, $[x]_0 = 5\,129.00\ldots$, $[x]_1 = 5\,129.30\ldots$ and so on. To avoid ambiguity, If x is a real number with two decimal expressions, $[x]_k$ will be the finite decimal built from the infinite decimal ending in 0's; for the number in Equation 0.5.1, $[x]_3 = 0.350$; it is not 0.349.

Given any two different finite numbers x and y, one is always bigger than the other. This is defined as follows. If x is positive and y is nonpositive, then $x > y$. If both are positive, then in their decimal expansions there is a left-most digit in which they differ; whichever has the larger digit in that position is larger. If both x and y are negative, then $x > y$ if $-y > -x$.

Page 19 The proof of Theorem 0.5.3 also fails to take into account the non-fractional part. The second paragraph of the proof should be replaced by the following:

Let the kth digit of a number be the digit in the 10^{-k} column. Thus if $k = -2$, the kth digit of 237.05 is 2; if $k = 0$, the kth digit is 7. Since $x \neq a$, there is then a smallest j such that $[x]_j < [a]_j$. There are 10 numbers that have the same kth digit as x for $k < j$ and that have 0 as the kth digit for $k > j$; consider those that are in $[\,[x]_j, a]$. This set is not empty, since $[a]_j$ is one of them. Let b_j be the largest such that $X \cap [b_j, a] \neq \phi$; such a b_j exists, since $x \in X \cap [[x]_j, a]$.

Now consider the set of numbers in $[b_j, a]$ that have the same kth digit as b_j for $k < j+1$, and 0 for $k > j+1$. Again this is a nonempty set with at most 10 elements, and b_j is one (the smallest) of them. Call b_{j+1} the largest such that $X \cap [b_{j+1}, a] \neq \phi$. Again such a b_{j+1} exists, since if necessary we can choose b_j. Keep going this way, defining numbers b_{j+2}, b_{j+3}, and so on, and let b be the number whose nth decimal digit (for all n) is the same as the nth decimal digit of b_k. We claim that $b = \sup X$.

Page 20 Last line of the proof of Theorem 0.5.7: $A - a_n \leq A - a_N$, not $A - a_n < A - a_N$.

Page 21 In the first line of the proof of Theorem 0.5.8, the summmand should be in parentheses: $\sum_{n=1}^{\infty}(a_n + |a_n|)$

Exercise 0.5.1: Exercise 1.6.11 repeats this exercise, with hints.

Page 23 On lines 7–8 we say that sets that can be put in one-to-one correspondence with the integers are called countable.

Two lines before Equation 0.6.4:

"of degree <= 2 with $|ai|$ <= 2", not " ... with ai <= 2".

At the very bottom of the page we say that "a set A is countable if $A \asymp \mathbb{N}$, and ... ". The natural numbers can be put in one-to-one correspondence with the integers, so there is no actual error, but in subsequent editions we will change "integers" to "natural numbers".

Page 24 Second full paragraph: $\mathcal{P}(E)$ is called the *power set* of E.

In the same paragraph, we use the symbol $\mapsto$, which is not explained until page 71. This can be avoided by some rewriting:

Clearly for any set E there exists a one-to-one map $f : E \to \mathcal{P}(E)$; for instance, the map $f(a) = \{a\}$.

Page 25 Exercise 0.6.6: Our solution does not actually use the hint. You can use Bernstein's theorem, but it seems a little harder than proving the result directly.

Page 31 Exercise 0.7.4: "Of the following complex ... ," not "of of ... ."

Page 32 Exercise 0.7.10: the word "number" should be plural.

Chapter 1

Page 40 In the margin note beginning "The trivial subspace," the **0** should be $\vec{\mathbf{0}}$.

Page 45 Sentence immediately before Definition 1.2.4: "I If" should be "If."

Page 48 Figure 1.2.5: On the right, the final matrix should be $A(BC)$, not $(AB)C$.

Page 52 The paragraph immediately before Example 1.2.21 should not be there; it was printed twice.

Page 57 Exercise 1.2.17 should read "Recall ... what the adjacency matrix of a graph is" (not "what the adjacency graph of a matrix is").

Page 61 In the first paragraph of the remark, the mention of "feedback" is incorrect. Feedback is compatible with linearity. The end of the paragraph, beginning with "Nor do linear transformations allow for feedback," has been rewritten as follows:

Nor does a linear model of the "price transformation" allow for the possibility that if you buy more you will get a discount for quantity, or that if you buy even more you might create scarcity and drive prices up. The failure to take such effects into consideration is a fundamental weakness of all models that linearize mappings and interactions.

Page 65 In Theorem 1.3.11, we assumed that the composition $T \circ S$ is a linear transformation. We should have stated this as part of the theorem and then proved it.

Thus the theorem should read

Theorem 1.3.11 (Composition corresponds to matrix multiplication). *Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^m \to \mathbb{R}^l$ are linear transformations given by the matrices $[S]$ and $[T]$ respectively. Then the composition $T \circ S$ is linear and*

$$[T \circ S] = [T][S]. \qquad 1.3.13$$

The proof should begin with a proof of linearity:

The following computation shows that $T \circ S$ is linear:

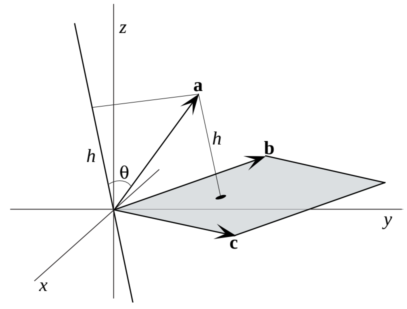
$$\begin{aligned}(T \circ S)(a\vec{\mathbf{v}} + b\vec{\mathbf{w}}) &= T\big(S(a\vec{\mathbf{v}} + b\vec{\mathbf{w}})\big) = T\big(aS(\vec{\mathbf{v}}) + bS(\vec{\mathbf{w}})\big)\\ &= aT\big(S(\vec{\mathbf{v}})\big) + bT\big(S(\vec{\mathbf{w}})\big) = a(T \circ S)(\vec{\mathbf{v}}) + b(T \circ S)(\vec{\mathbf{w}}).\end{aligned}$$

Page 71 Exercise 1.3.22 belongs in Section 1.4, as it uses the dot product and orthogonality.

Page 76 The proof of Theorem 1.4.5 should start with the sentence "If either $\vec{\mathbf{v}}$ or $\vec{\mathbf{w}}$ is $\mathbf{0}$, the statement is obvious, so suppose both are nonzero."

Page 79 Figure 1.4.9: $\mathbf{h}$ should be h, the height of the parallelogram.

Page 85 Figure 1.4.12: The arc is misplaced. It should go from $\mathbf{h}$ to $\mathbf{a}$, as shown below.



Page 93 In Equation 1.5.13, U should be $\overline{U}$.

Page 94 Third line: "an incontestably . . . ," not "a incontestably . . . ."

Page 98 Definition 1.5.20 of a limit of a function should have concerned a function $\mathbf{f}\colon X \to \mathbb{R}^m$. In the definition, the discussion, and the proof of Proposition 1.5.21, every f should be $\mathbf{f}$, a should be $\mathbf{a}$, and b should be $\mathbf{b}$.

Page 100 Theorem 1.5.23: We should have specified that U is a subset of $\mathbb{R}^n$. In Equation 1.5.35, we should have written $(h\mathbf{f})(\mathbf{x})$, not $h\mathbf{f}(\mathbf{x})$.

Page 101 Proof of Theorem 1.5.23: since $\mathbf{x}_0$ is not guaranteed to be in U, we should replace $\mathbf{f}(\mathbf{x}_0)$ by $\mathbf{a} := \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$ and $\mathbf{g}(\mathbf{x}_0)$ by $\mathbf{b} := \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}(\mathbf{x})$.

Page 101 The notation in the proof of Theorem 1.5.24 is a little mixed up. The theorem and proof should read

Theorem 1.5.24 (Limit of a composition). *Let $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ be subsets, and $\mathbf{f} : U \to V$ and $\mathbf{g} : V \to \mathbb{R}^k$ be mappings, so that $\mathbf{g}\circ\mathbf{f}$ is defined in U. If $\mathbf{y}_0 := \lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{f}(\mathbf{x})$ and $\mathbf{z}_0 := \lim_{\mathbf{y}\to\mathbf{y}_0} \mathbf{g}(\mathbf{y})$ both exist, then $\lim_{\mathbf{x}\to\mathbf{x}_0} \mathbf{g}\circ\mathbf{f}(\mathbf{x})$, exists, and*

$$\lim_{\mathbf{x}\to\mathbf{x}_0} (\mathbf{g}\circ\mathbf{f})(\mathbf{x}) = \mathbf{z}_0. \qquad 1.5.45$$

Proof. For all $\epsilon > 0$ there exists $\delta_1 > 0$ such that if $|\mathbf{y}-\mathbf{y}_0| < \delta_1$, then $|\mathbf{g}(\mathbf{y}) - \mathbf{z}_0| < \epsilon$. Next, there exists $\delta > 0$ such that if $|\mathbf{x}-\mathbf{x}_0| < \delta$, then $|\mathbf{f}(\mathbf{x}) - \mathbf{y}_0| < \delta_1$. Hence

$$|\mathbf{g}\big(\mathbf{f}(\mathbf{x})\big) - \mathbf{z}_0| < \epsilon \quad \text{when} \quad |\mathbf{x}-\mathbf{x}_0| < \delta. \quad \square \qquad 1.5.46$$

Page 102 Line 3: "the limit does not exist", not "the limit may not exist".

Page 103 Margin note, end of first paragraph: "for different values of $\mathbf{x}_0$", not "for different values of $\mathbf{x}$".

Theorem 1.5.28 (e): "even if $\mathbf{f}$ is not continuous at $\mathbf{x}_0$", not "even if $\mathbf{f}$ is not defined at $\mathbf{x}_0$".

Page 105 The proof of Proposition 1.5.34 should read

Set $\vec{\mathbf{a}}_i = \begin{bmatrix} a_{1,i} \\ \vdots \\ a_{n,i} \end{bmatrix}$. Then $|a_{k,i}| \le |\vec{\mathbf{a}}_i|$, so $\sum_{i=1}^{\infty} |a_{k,i}|$ converges, so by Theorem 0.5.8, $\sum_{i=1}^{\infty} a_{k,i}$ converges. Proposition 1.5.34 then follows from Proposition 1.5.13.

Last margin note: "Newton's method," not "Newton's method's."

Page 106 Third line of proof of Proposition 1.5.35: $S_k(I-A) = I - A^{k+1}$, not $S_l(I-A) = I - A^{k+1}$.

Note: One vigilant reader objected to Equation 1.5.60; how do we know that $\lim_{k\to\infty} S_k(I-A)$ exists? To be perfectly rigorous, we should have written the equation in the opposite direction, starting with $I = I - \lim_{k\to\infty} A^{k+1}$; then each step is justified.

Page 108 Exercise 1.5.12: We must assume $u(\epsilon) > 0$.

Page 109 Exercise 1.5.19 belongs with exercises for Section 1.6

Exercise 1.5.20, part (a): n was used with two different meanings. Below we changed n to k in two places:

(a) Let Mat (n, m) denote the space of $n \times m$ matrices, which we will identify with $\mathbb{R}^{nm}$. For what numbers $a \in \mathbb{R}$ does the sequence of matrices $A^k \in$ Mat $(2, 2)$ converge as $k \to \infty$, when $A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}$? What is the limit?

Pages 109–110 Exercise 1.5.21, part (d) and Exercise 1.5.24 are very similar.

Page 110: In the first line of Section 1.6, "convergence" should be "convergent".

Page 111 The subheading at the top of the page should be "The existence of a *convergent* subsequence in a compact set", not "the existence of a *convergence* subsequence . . . ".

Page 113 Six lines from the bottom, "the digits $0, 1, 1, 2, 4$" should be "the digits $0, 1, 2, 3, 4$"

Page 123 4th line from bottom: "of degree 1 or 2", not "of degree 1."

Page 125 Hint for Exercise 1.6.7: "minimum" not "maximum."

Page 126 It is incorrect to ascribe the motions of a pendulum to feedback.

Page 133 Equation 1.7.19: the 0 in $\lim \vec{\mathbf{h}} \to 0$ should be bold, since it is a vector.

Page 137 Equation 1.7.37 should have some parentheses:

$$\lim_{h\to 0} \frac{1}{h}\Big(\overbrace{(1+h)(1+2h)\sin(\frac{\pi}{2}+h)}^{\mathbf{f}(\mathbf{a}+h\vec{\mathbf{v}})} - \overbrace{(1\cdot 1\cdot \sin\frac{\pi}{2})}^{\mathbf{f}(\mathbf{a})}\Big).$$

Page 138 Remark: In several places we wrote $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ when we meant $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The $\vec{\mathbf{v}}$ in the expression $\Big[\mathbf{D}f \begin{bmatrix} 0 \\ 0 \end{bmatrix}\Big]\vec{\mathbf{v}}$ does not belong there. The last half of the remark should read:

. . . to a step of length $\sqrt{5}$ in the direction $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. To take a step of length 1 in that direction, starting at the origin, we would multiply $\Big[\mathbf{D}f \begin{bmatrix} 0 \\ 0 \end{bmatrix}\Big]$ by $\begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$, which has length 1, to get a rate of ascent (at time 0) of $19/\sqrt{5} \approx 8.5$. In which direction is the function increasing faster, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ or $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$?

In the footnote, $36/5 \approx 7.2$ should be $36/5 = 7.2$.

Pages 140 and 141 The last margin note on page 140 is almost identical to the first margin note on page 141.

Pages 143 and 145 The running heads at the top of the page should say "1.7 Differential Calculus," not "1.6 Four Big Theorems" and "1.8 Rules for Computing Derivatives."

Page 144 Line 4: "by direct computation", not "by direction computation".

Page 147 Last line: following the equation, we should perhaps add "i.e., for every $\vec{\mathbf{h}} \in \mathbb{R}^n$ we have $[\mathbf{Df}(\mathbf{a})]\vec{\mathbf{h}} = \mathbf{f}(\vec{\mathbf{h}})$."

Page 149 Long displayed equation after Equation 1.8.15, line 2: in the denominator at far right, $(f(\mathbf{a}))^2$ should be $(f(\mathbf{a}))$. In line 3, the notes in underbrackets could be more precise:

$$= \frac{1}{|\vec{\mathbf{h}}|}\left(\frac{f(\mathbf{a}) - f(\mathbf{a}+\vec{\mathbf{h}}) + [\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}}}{\big(f(\mathbf{a})\big)^2}\right) - \frac{1}{|\vec{\mathbf{h}}|}\left(\frac{f(\mathbf{a}) - f(\mathbf{a}+\vec{\mathbf{h}})}{\big(f(\mathbf{a})\big)^2} - \frac{f(\mathbf{a}) - f(\mathbf{a}+\vec{\mathbf{h}})}{f(\mathbf{a}+\vec{\mathbf{h}})\big(f(\mathbf{a})\big)}\right)$$

$$= \underbrace{\frac{1}{|\vec{\mathbf{h}}|}\left(\frac{f(\mathbf{a}) - f(\mathbf{a}+\vec{\mathbf{h}}) + [\mathbf{D}f(\mathbf{a})]\vec{\mathbf{h}}}{\big(f(\mathbf{a})\big)^2}\right)}_{\text{lim as } h\to 0 \text{ is 0 by def. of deriv.}} - \underbrace{\frac{1}{|\vec{\mathbf{h}}|}\frac{f(\mathbf{a}) - f(\mathbf{a}+\vec{\mathbf{h}})}{f(\mathbf{a})}}_{\text{bounded}}\underbrace{\left(\frac{1}{f(\mathbf{a})} - \frac{1}{f(\mathbf{a}+\vec{\mathbf{h}})}\right)}_{\text{lim as } h\to 0 \text{ is 0}}.$$

Page 149 Next to last margin note: "in all mathematics", not "in all of all mathematics".

Page 151 Equation 1.8.22: in the second matrix on the right side, the second entry in the third row should be 2, not 1:

$$\left[\mathbf{D}(\mathbf{f}\circ\mathbf{f})\begin{pmatrix}1\\1\\1\end{pmatrix}\right] = \begin{bmatrix}0 & 0 & 4\\ -2 & 1 & 0\\ 0 & 2 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & 2\\ 1 & 1 & 0\\ 0 & 2 & 0\end{bmatrix} = \begin{bmatrix}0 & 8 & 0\\ 1 & 1 & -4\\ 2 & 2 & 0\end{bmatrix}. \qquad 1.8.22$$

Page 154 Exercise 1.8.10: The function $\mathbf{f} : \mathbb{R}^2 \to \mathbb{R}^2$ should be the function $f : \mathbb{R}^2 \to \mathbb{R}$:

Page 157 Parentheses should be added to Equation 1.9.15:

$$\lim_{x\to 0}\left(\frac{1}{2} + 2x\sin\frac{1}{x}\right) = \frac{1}{2}.$$

Page 161 The = in the first line of Equation 1.9.25 should be $\leq$.
Exercise 1.9.1 is identical to Example 1.9.4.

Page 165 Exercise 1.22 is identical to Exercise 1.5.19, which in any case should be in Section 1.6.

Chapter 2

Page 171 Margin note next to Equation 2.1.4: $[A\vec{\mathbf{b}}]$ should be $[A\,|\,\vec{\mathbf{b}}]$. Bottom margin note: to be consistent with later notation, we should write $[A\,|\,\vec{\mathbf{b}}]$, not $[A\,,\,\vec{\mathbf{b}}]$ and $[A'\,|\,\vec{\mathbf{b}}']$, not $[A', \vec{\mathbf{b}}']$.

Page 172 Theorem 2.1.3: To be consistent with later notation, we should write $[A\,|\,\vec{\mathbf{b}}]$, not $[A\,,\,\vec{\mathbf{b}}]$ and $[A'\,|\,\vec{\mathbf{b}}']$, not $[A', \vec{\mathbf{b}}']$.
First margin note: $[A\,|\,\tilde{\mathbf{b}}]$, not $[A\tilde{\mathbf{b}}]$
We use the vertical line to avoid confusion with the *product* $A\vec{\mathbf{b}}$. You should not think that $\vec{\mathbf{b}}$ is somehow special as far as row reduction is concerned; the rules of row reduction apply equally to all the columns of $[A\,|\,\vec{\mathbf{b}}]$: the columns of A and the column $\vec{\mathbf{b}}$.

Page 176 Exercise 2.1.5, "in the algorithm for row reduction" should be "in Definition 2.1.1 of row operations".

Page 178 Margin note: The vector $\vec{\mathbf{b}}$ does not contain the solutions.

Pages 178, 179, 181 As for Page 172, to keep notation consistent, various augmented matrices should have vertical lines, not commas, as in $[\widetilde{A}\,|\,\widetilde{\vec{\mathbf{b}}}]$.

Page 179 In the second line of Theorem 2.2.4, the $\mathbf{x}$ should be $\vec{\mathbf{x}}$: $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, not $A\mathbf{x} = \vec{\mathbf{b}}$.

In the remark, we mention linear independence prematurely; it is not discussed until Section 2.4.

Page 181 In the proof of Theorem 2.2.4, the $\mathbf{x}$ should be $\vec{\mathbf{x}}$: $A\vec{\mathbf{x}} = \vec{\mathbf{b}}$, not $A\mathbf{x} = \vec{\mathbf{b}}$.

Page 185 Exercise 2.2.6, part (a) should read: "For what values of a does the system of equations in the margin have a solution?" (not "have a unique solution").

Page 188 Part (3) of Definition 2.3.6: "$i \neq j$", not $1 \neq j$.

Page 195 Definition 2.4.5 was perhaps not clear. Here is a rewrite:

Definition 2.4.5 (Linear independence). The vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k \in \mathbb{R}^n$ are linearly independent if every vector in $\mathbb{R}^n$ can be written as a linear combination of $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ in at most one way, i.e.:

$$\sum_{i=1}^{k} x_i \vec{\mathbf{v}}_i = \sum_{i=1}^{k} y_i \vec{\mathbf{v}}_i \quad \text{implies} \quad x_1 = y_1,\ x_2 = y_2, \dots, x_k = y_k.$$

Page 196 First margin note, 4th line after the matrices: "is upper triangular with nonzero entries ... ", not "is upper triangular form with ... ".

Page 205 Exercise 2.4.11 should be with the exercises for Section 2.5.

Page 206 Part (a) of Exercise 2.4.13 was poorly stated. It should be:

(a) For $n = 1, n = 2, n = 3$, write the system of linear equations which the $a_{0,n}, \dots, a_{n,n}$ must satisfy so that the integral of 1 is exact, the integral of x is exact, and so on, until you get to x^n.

Exercise 2.5.14, part (c): W should be W_t.

Page 212 Corollary 2.5.11: Rather than "i.e., if the kernel is zero" it would be better to say, "i.e., if the kernel has dimension 0."

Page 214 In the second box (giving equivalent statements about a one-to-one linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$), statement 6 is incorrect. It should be:

The row-reduced matrix $\widetilde{A}$ has no nonpivotal column.

Page 222 The parts of Exercise 2.5.7 are listed as (a), (b), (c), (b). Of course the second (b) should be (d).

Page 224 Part (c) of Exercise 2.5.20: "For any vectors $\vec{\mathbf{b}} \in \mathbb{R}^n$", not "for any numbers $\vec{\mathbf{b}} \in \mathbb{R}^n$".

There is also an extra period in the margin note.

Page 226 Fourth line of Example 2.6.3: a space is needed between "Example 2.6.2" and "and."

Page 231 A plus sign is missing from Equation 2.6.18. It should be

$$\mathbf{v'}_i = p_{1,i}\mathbf{v}_1 + p_{2,i}\mathbf{v}_2 + \cdots + p_{n,i}\mathbf{v}_n.$$

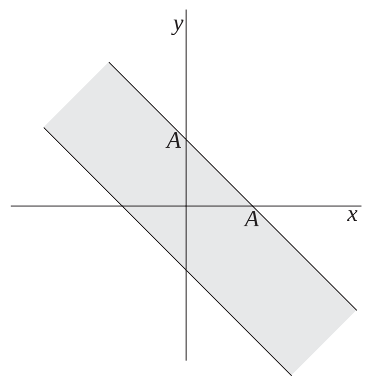
Page 236 Exercise 2.6.3: The four matrices do not form a basis, since $\underline{\mathbf{v}}_3 = -\underline{\mathbf{v}}_4$.

Exercise 2.6.5: After the displayed equation, $\Phi^{-1}_{\{\underline{\mathbf{v}}\}}$ should be $\Phi_{\{\underline{\mathbf{v}}\}}$: "so that $\Phi_{\{\underline{\mathbf{v}}\}} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$."

Page 237 Exercise 2.6.11: A_a should be A.

Page 243 In Definition 2.7.4 we should have required that $U \subset \mathbb{R}^n$ be open.

Page 244 The way Figure 2.7.5 is drawn, the shaded strip seems to end in quadrants 2 and 4; actually, it is infinite. The following may suggest reality better:



Page 246 In Example 2.7.11, we use a different order for the subscripts of c than that given in Proposition 2.7.10. To make the text consistent, $c_{2,2,1}$ should be $c_{1,2,2}$ and $c_{1,1,2}$ should be $c_{2,1,1}$:

$$|D_2D_2\mathbf{f}_1| \leq 3A = \underbrace{c_{1,2,2}}_{\substack{\text{bound for} \\ |D_2D_2\mathbf{f}_1|}} \quad \text{and} \quad |D_1D_1\mathbf{f}_2| \leq 3A = \underbrace{c_{2,1,1}}_{\substack{\text{bound for} \\ |D_1D_1\mathbf{f}_2|}}$$

with all others 0, so

$$\sqrt{c^2_{1,2,2} + c^2_{2,1,1}} = 3A\sqrt{2}. \qquad 2.7.40$$

Page 246 Four lines from the bottom– one reader wondered whether "blunderbuss" was "a new word from generation X". Our dictionary defines a blunderbuss as an "old-fashioned, short gun with large bore and flaring mouth, used for scattering shot at close range". It will hit a big target, but is not precise.

Page 249 Statement of Theorem 2.7.13: in the next-to-last line, it should be "has a unique solution in the closed ball $\overline{U_0}$". To see why this is necessary, consider Example 2.8.1, where Newton's method converges to 1, which is not in U_0 but is in its closure.

In the bottom margin note, we discuss the importance of making sure both sides of an equation have the same units. In chemical engineering, fluid mechanics, etc., this is called "dimensional analysis."

Page 250 At the end of Equation 2.7.55 we should write < 1.2, not < 2:

$$\left|[\mathbf{D}\vec{F}(\mathbf{a}_0)]^{-1}\right|^2 = \frac{1}{(\cos 2 - 1)^2}\Big((\cos 2)^2 + 1 + (1 - \cos 2)^2\Big) \sim 1.1727 < 1.2, \quad 2.7.55$$

Equation 2.7.56 contains several errors. In the second rows of the matrices on the right, two minus signs should be pluses. In the third line of Equation 2.7.56, the first $=$ in the last line should be $\leq$, the 4 under the square root should be 8, and the 2 after the second $=$ should be $2\sqrt{2}$.

The equation should be:

$$\begin{aligned}\left|\left[\mathbf{D}\vec{F}\begin{pmatrix}x_1\\y_1\end{pmatrix}\right] - \left[\mathbf{D}\vec{F}\begin{pmatrix}x_2\\y_2\end{pmatrix}\right]\right| &= \left|\begin{bmatrix} -\sin(x_1-y_1)+\sin(x_2-y_2) & \sin(x_1-y_1)-\sin(x_2-y_2) \\ \cos(x_1+y_1)-\cos(x_2+y_2) & \cos(x_1+y_1)-\cos(x_2+y_2)\end{bmatrix}\right| \\ &\leq \left|\begin{bmatrix} |-(x_1-y_1)+(x_2-y_2)| & |(x_1-y_1)-(x_2-y_2)| \\ |(x_1+y_1)-(x_2+y_2)| & |(x_1+y_1)-(x_2+y_2)|\end{bmatrix}\right| \\ &\leq \sqrt{8\big((x_1-x_2)^2+(y_1-y_2)^2\big)} = 2\sqrt{2}\left|\begin{pmatrix}x_1\\y_1\end{pmatrix} - \begin{pmatrix}x_2\\y_2\end{pmatrix}\right|. \quad 2.7.56\end{aligned}$$

(Going from the second to the third line of Equation 2.7.56 uses $(a+b)^2 \leq 2(a^2+b^2)$.)

Page 251 In the first line, $M = 2\sqrt{2}$, not $M = 2$.

Equation 2.7.57 should be:

$$|\vec{F}(\mathbf{a}_0)|\left|[\mathbf{D}\vec{F}(\mathbf{a}_0)]^{-1}\right|^2 M \leq .1 \cdot 1.2 \cdot 2\sqrt{2} \approx .34 < .5. \quad 2.7.57$$

Page 253 As on page 246, the order of subscripts for c is wrong in three places at the bottom of the page. Below, the starred entries have been corrected:

$$\begin{array}{ll} \sup|D_1D_1f_1| \leq 3 = c_{1,1,1} & *\sup|D_1D_1f_2| = 0 = c_{2,1,1} \\ \sup|D_1D_2f_1| \leq 1 = c_{1,2,1} & *\sup|D_1D_2f_2| = 0 = c_{2,2,1} \\ *\sup|D_2D_2f_1| \leq 1 = c_{1,2,2} & \sup|D_2D_2f_2| = 2 = c_{2,2,2}. \end{array}$$

Page 255 Exercise 2.7.3 involves showing that a function is Lipschitz, but we did not actually define a Lipschitz function in the text. If $X \subset \mathbb{R}^n$, then a mapping $\mathbf{f} : X \to \mathbb{R}^m$ is Lipschitz if there exists C such that

$$\big|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})\big| \leq C|\mathbf{x} - \mathbf{y}|.$$

(Of course a Lipschitz mapping is continuous; it is better than continuous.)

Page 256 The margin note about Exercise 2.23 belongs on page 288.

Exercise 2.7.11 is missing part (b): Prove that this Newton's method converges.

Page 260 There should be no vector $\vec{\mathbf{v}}$ in Definition 2.8.6. The definition should read

Definition 2.8.6 (The norm of a linear transformation). Let $A : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The norm $\|A\|$ of A is

$$\|A\| = \sup |A\vec{\mathbf{x}}|, \text{ when } \mathbf{x} \in \mathbb{R}^n \text{ and } |\vec{\mathbf{x}}| = 1. \qquad 2.8.11$$

Page 261 On the second line of Example 2.8.9 we say that the norm is $\frac{1+\sqrt{5}}{2}$; in Equation 2.8.8 we compute the norm as $\sqrt{\frac{3+\sqrt{5}}{2}}$. Both, of course, are correct, since

$$\sqrt{\frac{3+\sqrt{5}}{2}} = \sqrt{\frac{6+2\sqrt{5}}{4}} = \frac{1+\sqrt{5}}{2}.$$

Page 264 Exercise 2.8.8: In the displayed equation, D should be D^2:

$$\|A\| = \left(\frac{|A|^2 + \sqrt{|A|^4 - 4D^2}}{2}\right)^{1/2}.$$

Fourth line from bottom: "mainly" should be "namely".

Page 265 First sentence after Theorem 2.9.2: Exercise A.7.1, not 7.1.

Page 270 We never proved Equation 2.9.13! Moreover, it is wrong, which shows how dangerous it is to omit proofs. The correct equation is

$$R_1 \;=\; R|L^{-1}|^2 \left(\sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}} - |L|\right).$$

Proof. Suppose $|\mathbf{x} - \mathbf{x}_0| < R_1$. Then

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| \;\le\; |\mathbf{x} - \mathbf{x}_0| \sup |[\mathbf{Df}(\mathbf{x})]| \;\le\; R_1 \sup |[\mathbf{Df}(\mathbf{x})]|.$$

We find a bound for $|[\mathbf{Df}(\mathbf{x})]|$:

$$|[\mathbf{Df}(\mathbf{x})] - [\mathbf{Df}(\mathbf{x}_0)]| = |[\mathbf{Df}(\mathbf{x})] - L| \underbrace{\le}_{\text{Eq. 2.9.11}} \frac{1}{2R|L^{-1}|^2} |\mathbf{x} - \mathbf{x}_0| \le \frac{R_1}{2R|L^{-1}|^2}$$

so

$$|[\mathbf{Df}(\mathbf{x})]| \le |L| + \frac{R_1}{2R|L^{-1}|^2}, \quad \text{i.e.,} \quad \sup |[\mathbf{Df}(\mathbf{x})]| = |L| + \frac{R_1}{2R|L^{-1}|^2}.$$

Therefore (remember that R is the radius of V, the domain of $\mathbf{g}$) we want to find the largest R_1 satisfying

$$R \;\ge\; \left(|L| + \frac{R_1}{2R|L^{-1}|^2}\right) R_1.$$

The right-hand side is 0 when $R_1 = 0$ and then increases as R_1 increases, so we want the largest value of R_1 for which the inequality is an equality. Thus we want to solve the quadratic equation

$$R_1^2 + 2R|L^{-1}|^2\,|L|R_1 - 2R^2|L^{-1}|^2 = 0,$$

which gives

$$R_1 = R|L^{-1}|^2 \left(-|L| + \sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}} \right).$$

Page 271 At the end of the first paragraph after Figure 2.9.6: "look at condition (3a) of the theorem" should be "look at condition (1) of the theorem." In the next paragraph, "condition (3b) is more delicate" should be "condition (2) is more delicate."

Page 271 Last line: "an inverse function," not "a inverse function."

Page 273 In the first line after Equation 2.9.18, the reference to Equation 2.9.24 should be to Equation 2.9.11.

Pages 277, 278 The margin note about Equation 2.9.30 (page 277) should be on page 278.

Page 280 The second margin note is completely false; we have no idea what we were thinking of. Using the second partial derivative method in Example 2.9.15 is perfectly possible and gives a Lipschitz ratio of $2\sqrt{3}$.

Page 281 Second line: "diagonal matrices" should be "diagonal entries."

Page 284 In Execise 2.9.4, the matrix $\begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix}$ should be $\begin{bmatrix} -3 & 0 \\ 0 & -3 \end{bmatrix}$. This matrix appears three times.

Page 286 The parts of Exercise 2.11 are mislabeled. They should be (a), (b), (c).

Page 288 The margin note on page 256 about Exercise 2.23 belongs on this page.

Page 290 Exercise 2.33: in two places, "of degree" should be "of degree at most":

"q_1 and q_2 are polynomials of degrees at most $k_2 - 1$ and $k_1 - 1$"

and

"the space of polynomials of degree at most $k_1 + k_2 - 1$."

(Say we have a polynomial $ax^2 + bx + c$. For it to live in a vector space, we have to allow for the possibility that $a = 0$. But then it is a first degree polynomial.) For the second sentence of the exercise, where we discuss p_1 and p_2, we don't have to say "at most" because those are specific polynomials. But q_1 and q_2 are variables.

Chapter 3

Page 294 Caption to Figure 3.1.3. We wrote that "the curve in $I_1 \times J_1$ can also be thought of as the graph of a function expressing $x \in I_1$ as a function of $y \in J_1$", but this is wrong, because we made J_1 too big; there are values of $y \in J_1$ that give no values in $I - 1$.

Page 297 Third line: "variables in terms," not "variables terms."

Page 300 There are several mistakes in margin notes. In the second line of the first note, $[\mathbf{D}F(\mathbf{a})]$should be $[\mathbf{DF}(\mathbf{a})]$. The next note has the reverse problem: the $\mathbf{F}$ should be F; in the case of a curve in the plane, $\mathbb{R}^{n-k} = \mathbb{R}$ and $\mathbf{F}$ is the single function F. The last note is wrong; it should read, "More generally, for an $(n-1)$-dimensional manifold in any $\mathbb{R}^n$, ... "

Page 305 Line 5 should read: " ... good picture of a parametrized curve or surface" not " ... good picture of parametrized the curve or surface."

Page 307 (comment, not correction) Example 3.1.17: In this interpretation, $\gamma'(t)$ is the *velocity vector*; it is tangent to the curve at $\gamma(t)$ and its length is the speed at which you are traveling at time t.

Page 307 Five lines from the bottom, "The requirement that $[\mathbf{D}\gamma(\mathbf{u})]$ be one" should be "The requirement that $[\mathbf{D}\gamma(\mathbf{u})]$ be one to one," and

$$\vec{\gamma}'(t) \neq \mathbf{0}'' \quad \text{should be} \quad \vec{\gamma}'(t) \neq \mathbf{0}.$$

Page 308 2nd line of the remark: "may look as though", not "may looks as though".

Page 315 Exercise 3.1.24, 2nd line: "a smooth curve", not "is a smooth curve".

Exercise 3.1.26: we used A both to denote the $A(n,n)$ (the space of antisymmetric $n \times n$ matrices) and to denote the matrice A. The matrix A is $n \times n$.

Page 316 Exercise 3.1.28, part (c): $\mathbf{g}$, not g.

Definition 3.2.1 is not stated correctly. It should be

Definition 3.2.1 (Tangent space of a manifold). Let $M \subset \mathbb{R}^n$ be a k-dimensional manifold, so that near $\mathbf{z} \in M$, M is the graph of a C^1 mapping $\mathbf{f}$ expressing $n-k$ variables as functions of the other k variables. If $\mathbf{z} = \mathbf{a}+\mathbf{f}(\mathbf{a})$, then the tangent space to M at $\mathbf{z}$, denoted $T_{\mathbf{z}}M$, is the graph of $[\mathbf{Df}(\mathbf{a})]$.

What does $\mathbf{z} = \mathbf{a} + \mathbf{f}(\mathbf{a})$ mean? The point $\mathbf{a}$ is in a k-dimensional subset of $\mathbb{R}^n$; it has n entries but $n-k$ of them are 0. Similarly, $\mathbf{f}(\mathbf{a})$ has n entries but k of them are 0. So we are adding two n-dimensional points to get a third n-dimensional point. What makes us slightly uneasy is that we aren't supposed to add points, just vectors. We would prefer to write $\mathbf{z} = \begin{pmatrix} \mathbf{a} \\ \mathbf{f}(\mathbf{a}) \end{pmatrix}$ but that would be assuming that the k "active" variables comes first, which isn't necessarily the case.

Page 317 The title to Example 3.2.2 should be "Tangent line and tangent space to smooth curve", not "Tangent line and tangent plane ... ".

Page 318 Example 4.2.3, second line of second paragraph: "playing the role of $\mathbf{x}$" (not $\mathbf{x}_1$).

Page 319 Example 3.2.5 refers to Example 3.1.11, but that example concerned a different function. Example 3.2.5 has been rewritten to show that X_c is a smooth curve for all c:

The locus X_c defined by $x^9+2x^3+y+y^5=c$ is a smooth curve for all values of c since the derivative of the function $F\begin{pmatrix} x \\ y \end{pmatrix} = x^9+2x^3+y+y^5$ is

$$\left[\mathbf{D}F\begin{pmatrix} x \\ y \end{pmatrix}\right] = [9x^8+6x^2\,,\,1+5y^4],$$

and $1+5y^4$ is never 0.

Page 320 We should have said that $\mathbf{a} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

Page 321 Equation 3.2.19 should end with $= \mathbf{0}$:

$$[D_{j_1}\mathbf{F}(\mathbf{c}),\dots,D_{j_{n-k}}\mathbf{F}(\mathbf{c})]\dot{\mathbf{x}} + [D_{i_1}\mathbf{F}(\mathbf{c}),\dots,D_{i_k}\mathbf{F}(\mathbf{c})]\dot{\mathbf{y}} = \mathbf{0}.$$

Page 322 Exercise 3.2.5: parts (a) and (b) not (a) and (c)

Page 325 In the first line after Equation 3.3.10, the reference should be to Equation 3.3.9 and footnote 7.

Page 326 First line after Equation 3.3.16: "There are 30 such terms" refers to terms *other* than the five terms in Equation 3.3.16. Thus there are 35 in all.

Page 332 The first term in the 4th line should have a minus sign:

$$-4y\sin(x+y^2).$$

Line immediately before Equation 3.3.38: "$(-\frac{1}{3}!)h_1^3$" should be "$(-\frac{1}{3!})h_1^3$."

Page 334 Exercise 3.3.6: in part (b), the hypothesis $f(-\mathbf{x}) = -f(\mathbf{x})$ should have been included.

Page 335 Exercise 3.3.14: It's possible to solve this using partial derivatives (and a computer), but it's much easier with the techniques of Section 3.4; the exercise should be with the exercises for that section.

Page 336 In the main text, Edmund Landau's dates are given incorrectly. They are correct in the margin note.

Lines 2 and 3 from bottom: "We will write them only near 0, but by translation they can be written at any point where the function is defined" (not " ... they can be written anywhere").

Page 337 The first margin note suggests, incorrectly, that all odd functions and all even functions have Taylor polynomials. It should read

"The Taylor function of an odd function can have only odd terms, and the Taylor function of an even function can have only even terms."

Page 340 Equation 3.4.17 should include $= 0$:

$$F\begin{pmatrix} x \\ y \end{pmatrix} = x^3+xy+y^3-3=0.$$

Page 342 Exercise 3.4.4 should read

Find numbers a, b, c such that when f is C^3,

$$h\Big(af(0)+bf(h)+cf(2h)\Big) - \int_0^h f(t)\,dt \in o(h^3).$$

(If you omit the factor h, then a, b, c are not numbers, but multiples of h.)

Exercise 3.4.5 should read

Find numbers a, b, c such that when f is C^3,

$$h\Big(af(0) + bf(h) + cf(2h)\Big) - \int_0^{2h} f(t)\,dt \in o(h^3).$$

Page 343 In the third line, the equality sign should be raised: "But $p(\mathbf{x}) = x_1x_2x_3$" (not But $p(\mathbf{x})_= x_1x_2x_3$").

In the displayed equation in the margin, $Q(t)$ should be $Q(f)$:

$$Q(f) = \int_0^1 (f(x))^2\,dx.$$

The bottom margin note is incorrect; the theorem is due to Fermat but it is not Fermat's little theorem.

Page 345 Equation 3.5.6 should have a "plus or minus":

$$\sqrt{a}x + \frac{b}{2\sqrt{a}} = \pm\sqrt{\frac{b^2 - 4ac}{4a}}.$$

Page 347 The margin note halfway down the page should specify a quadratic form on $\mathbb{R}^n$:

"Definition 3.5.9 is equivalent to saying that a quadratic form on $\mathbb{R}^n$ is positive definite if its signature is $(n, 0)$ and negative definite if its signature is $(0, n)$."

A quadratic form on $\mathbb{R}^n$ with signature $(k, 0)$, $k < n$, is not positive definite.

Margin note beginning "Definition 3.5.9 is equivalent": Exercise 3.5.7 concerns only positive definite quadratic forms.

The last margin note, about $Q(p)$, should be on page 343.

Pages 348–349 In several places – Equations 3.5.23, 3.5.25, and 3.5.28, and in the sentence before Equation 3.5.25 – we write things of the form $\alpha_1(\mathbf{x})^2$ which would be better written with an additional set of parentheses: $(\alpha_1(\mathbf{x}))^2$.

Page 351 Exercise 3.5.1: "and finally the terms in x," not " ... in y."

Page 356 Last margin note: This is true for a quadratic form on $\mathbb{R}^n$.

Page 357 First margin note: This is true for a quadratic form on $\mathbb{R}^n$.

Page 360 Exercise 3.6.5, part (a) should read

(a) Find the critical points of the function $f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = xy + yz - xz + xyz$.

Page 362 There should be a $\triangle$ to mark the end of Example 3.7.3.

Page 363 The caption to Figure 3.7.2: unnecessary comma in the first line.

Page 365 The $\triangle$ at the end of the caption should be on the next page, at the end of the example.

Page 366 Third paragraph of Example 3.7.6: The reference should be to Definition 3.1.16, not 3.1.18.

Page 367 Margin note immediately after the figure caption: "manifold," not "manifolds."

Page 368 Example 3.7.9 contains various errors. It should read as follows:

Example 3.7.9 (Critical points of functions constrained to ellipse). Let us follow this procedure for the function $f(\mathbf{x}) = x^2 + y^2 + z^2$ of Example 3.7.4, constrained as before to the ellipse given by

$$\mathbf{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 + y^2 - 1 \\ x - z \end{pmatrix} = \mathbf{0}. \qquad 3.7.19$$

We have

$$[Df(a)] = [2x, 2y, 2z],\ [DF_1(a)] = [2x, 2y, 0],\ [DF_2(a)] = [1, 0, -1],$$

so Theorem 3.7.7 says

$$[2x, 2y, 2z] = \lambda_1[2x, 2y, 0] + \lambda_2[1, 0, -1], \qquad 3.7.20$$

which gives

$$2x = \lambda_1 2x + \lambda_2, \quad 2y = \lambda_1 2y + 0, \quad 2z = \lambda_1 \cdot 0 - \lambda_2. \qquad 3.7.21$$

If $y \neq 0$, this gives $\lambda_1 = 1$, $\lambda_2 = 0$, and $z = 0$. The equations $F_1 = 0$ and $F_2 = 0$ then say that $x = 0$, $y = \pm 1$, so $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ are critical points. But if $y = 0$, then $F_1 = 0$ and $F_2 = 0$ give $x = z = \pm 1$. So $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$, with $\lambda_1 = 2$ and $\lambda_2 = \mp 2$, are also critical points.

Since our constraint is a compact manifold, the maximum and minimum values of f restricted to $\mathbf{F} = \mathbf{0}$ are attained at constrained critical points of f. Since $f\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = f\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} = 1$ and $f\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = f\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix} = 2$, we see that f achieves its maximum value of 2 at $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and at $\begin{pmatrix} -1 \\ 0 \\ -1 \end{pmatrix}$ and its minimum value of 1 at $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$. △

Page 370 There should be a △ to mark the end of Example 3.7.11.

Page 372 First paragraph: "the principal axis theorem," not "the principle axis theorem."

Page 374 Theorem 3.7.16 should read

Theorem 3.7.16. *A quadratic form Q_A has signature (k, l), if and only if A has k linearly independent eigenvectors with positive eigenvalues and l linearly independent eigenvectors with negative eigenvalues.*

The last margin note should refer to Equation 3.7.55 (not 3.7.54) and should be on the next page.

Page 375 Last margin note: The hint is for both parts of Exercise 3.7.7, not just part (b).

Page 376 Exercise 3.7.8: For $a, b \geq 0$.

Exercise 3.7.11: the parts are mislabeled: (d) should be (c), etc.

Exercise 3.7.13: "closest to and furthest from", not "closest and furthest from".

Exercise 3.7.14: We should have said "the unit circle".

Page 378 Equation 3.8.5: $g(X)$, not $g(x)$.

Page 379 Example 3.8.3, next to last line: the curvature $\frac{2}{5\sqrt{5}}$ is about 0.179, not 0.896. (We had put the $\sqrt{5}$ in the numerator.) We have redone Figure 3.8.2 below.

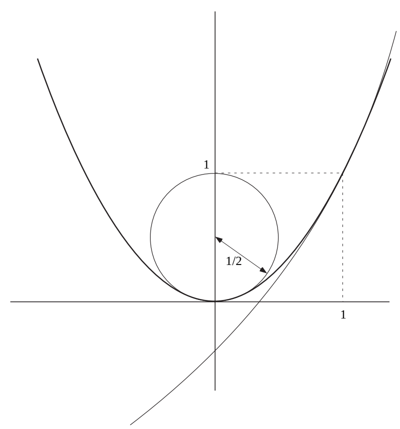


FIGURE 3.8.2. At $\begin{pmatrix}1\\1\end{pmatrix}$, which corresponds to $a = 1$, the parabola given by $y = x^2$ looks much flatter than the unit circle. Instead, it resembles a circle of radius $5\sqrt{5}/2 \approx 5.59$. (A portion of such a circle is shown. Note that it crosses the parabola. This is the usual case, occurring when, in adapted coordinates, the cubic terms of the Taylor polynomial of the difference between the circle and the parabola are nonzero.) At the origin, which corresponds to $a = 0$, it has curvature 2 and resembles the circle of radius 1/2, which also has curvature 2. "Resembles" is an understatement. At the origin, the Taylor polynomial of the difference between the circle and the parabola starts with fourth-degree terms.

Page 383 The sentence following Definition 3.8.8 should be: Exercise 3.8.3 asks you to show that the absolute value of the mean curvature of the unit sphere is 1 and that the Gaussian curvature of the unit sphere is 1.

Page 384 Caption for Figure 3.8.6: The discussion of the second and third goats should read

"The second goat is thin. He lives on the top of a hill, with positive Gaussian curvature; he can reach less grass. The third goat is fat. His surface has negative Gaussian curvature; with the same length chain, he can get at more grass. This would be true even if the chain were so heavy that it lay on the ground."

Page 385 Proposition 3.8.9 does not apply to a surface known in "best" coordinates, where the Taylor polynomial starts with quadratic terms; in that case the linear terms a_1 and a_2 would be 0.

Page 387 Equation 3.8.42: The numerator should be $4(a^2 - b^2)$.

Page 392 The fourth line of Equation 3.8.68 should be

$$= \Big(-\big(\kappa\big(s(t)\big)\big)^2\big(s'(t)\big)^3 + s'''(t)\Big)\vec{\mathbf{t}}(s(t))$$

For consistency, the last line of Equation 3.8.68 should be $\vec{\mathbf{b}}\big(s(t)\big)$, not $\vec{\mathbf{b}}$.

Page 393 Exercise 3.8.3: show that the absolute value of the mean curvature of the unit sphere is 1 and that the Gaussian curvature is 1.

Page 394 In the hint for Exercise 3.8.11, we neglected to define $SO(3)$. It is the space of orthogonal 3×3 matrices with determinant $+1$. (Recall that an orthogonal $n \times n$ matrix is a matrix whose columns form an orthonormal basis of $\mathbb{R}^n$.)

Page 398 Exercise 3.21, part (a): $2d\cos\varphi$ should be $2ad\cos\varphi$:

$$a^2 + d^2 - 2ad\cos\varphi \;=\; b^2 + c^2 - 2bc\cos\psi.$$

Chapter 4

Page 402 Six lines from the bottom, in the statement labeled (2): "exists", not "exits".

Page 411 We should have included in this section the following statement about how volume scales, for an arbitrary subset of $\mathbb{R}^n$:

Proposition (Scaling volume). *If $A \subset \mathbb{R}^n$ has volume and $t \in \mathbb{R}$, then tA has volume and* $\mathrm{vol}_n(tA) = t^n \,\mathrm{vol}_n(A)$.

Proof. By Proposition 4.1.19, this is true if A is a parallelogram, in particular if A is a cube $C \in \mathcal{D}_N$. Assume A is any subset of $\mathbb{R}^n$. For any N, let f_N be the function that is the constant function 1 on cubes in $\mathcal{D}_N$ that are completely inside A, and let g_N be the function that is the constant function 1 on cubes in $\mathcal{D}_N$ that completely cover A:

$$f_N = \sum_{\substack{C \in \mathcal{D}_N,\\ C \subset A}} \chi_C, \quad g_N = \sum_{\substack{C \in \mathcal{D}_N,\\ C \cap A \neq \emptyset}} \chi_C,$$

so that $f_N \le \chi_A \le g_N$. Then

$$f_N(t\mathbf{x}) \le \chi_A(t\mathbf{x}) = \chi_{tA}(\mathbf{x}) \le g_N(t\mathbf{x}).$$

By Proposition 4.1.19,

$$\int f_N(t\mathbf{x}) = \sum_{\substack{C\in\mathcal{D}_N,\\ C\subset A}} \int \chi_C(t\mathbf{x}) = \sum_{\substack{C\in\mathcal{D}_N,\\ C\subset A}} \overbrace{\int \chi_{tC}(\mathbf{x})}^{\mathrm{vol}_n\, tC} \underbrace{=}_{\text{Prop.4.1.19}} t^n \sum_{\substack{C\in\mathcal{D}_N,\\ C\subset A}} \overbrace{\int \chi_C(\mathbf{x})}^{\mathrm{vol}_n\, C} = t^n \int f_N(\mathbf{x}).$$

(We omitted the $|d^n\mathbf{x}|$ in the integrals above in hopes of making the equation more readable.) Similarly, $\int g_N(t\mathbf{x}) = t^n \int g_N(\mathbf{x})$. Thus

$$t^n \int f_N(\mathbf{x}) = \int f_N(t\mathbf{x}) \le L(\chi_{ta}) \le U(\chi_{tA}) \le \int g_N(t\mathbf{x}) = t^n \int g_N(\mathbf{x}).$$

Since A has volume,

$$\lim_{N\to\infty} t^n \int f_N = \lim_{N\to\infty} t^n \int g_N = t^n \,\mathrm{vol}_n\, A,$$

so, in particular, for any $\epsilon > 0$,

$$U(\chi_{tA}) - L(\chi_{tA}) < \epsilon,$$

so $U(\chi_{tA}) = L(\chi_{tA})$, so χ_{tA} is integrable, and

$$\mathrm{vol}_n(tA) = \int \chi_{tA} = t^n \,\mathrm{vol}_n(A).$$

□

Page 412 In Exercise 4.1.5, part (d) and Exercise 4.1.6, part (c), a should be positive: $0 < a < b$.

Page 413 Exercise 4.1.14: Since the geometric mean for negative numbers is problematic, it would be better to define f as follows:

$$f(x) = \begin{cases} 0 & \text{if } x \notin [0,1], \text{ or } x \text{ is rational} \\ 1 & \text{if } x \in [0,1], \text{ and } x \text{ is irrational.} \end{cases}$$

Page 416 Note: The statement that an outcome with probability 0 will not occur may seem to contradict the statement, in the subsequent discussion of infinite, continuous sample spaces, that in such a setting "each individual outcome has probability 0." There is actually no contradiction. When a sample space is infinite, an individual outcome cannot occur because it is physically meaningless. We can think of spinning a bottle so that it ends up at exactly angle $\pi/2$, but we could never measure such a result. So, although it may seem obvious that each time we spin the bottle it lands on some angle, we really should think of it as landing within some measurable range of angles. It may seem peculiar that an infinite number of outcomes each with probably 0 can add up to something positive (in this case, 2π), but it is the same as the more familiar notion that a line has length, while the points that compose it have length 0.

Page 417 Margin note, third line from the bottom: "introducing them".

Page 418 2nd line after Equation 4.2.9: "intersects," not "intersect."

Line above Equation 4.2.10: $\int_0^\pi \sin\theta\, |d\theta| = 2$. (It does not equal π.)

Page 419 Margin note, line 4: “to 20 feet,” not “to20 feet.”

Page 420 Line 8: there is an extra period after “data.”

Page 421 Equation 4.2.15: this sums to 2, not 4/3! So in the next sentence, it should be “any sum smaller than $2 … ”

We get the result 2 as follows:

For $|x| < 1$, we have

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x}, \quad \text{so} \quad \sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2},$$

which gives

$$\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2},$$

which gives

$$\sum_{n=1}^{\infty} n \cdot \frac{1}{2^n} = \frac{1/2}{(1/2)^2} = 2.$$

Page 422 Definition 4.2.12 of variance: There is an unfortunate typo in Equation 4.2.17; a $\mu(\mathbf{x})$ was omitted on the right-hand side. The equation should be

$$\operatorname{Var}(f) = E\Big(\big(f - E(f)\big)^2\Big) = \int_S \big(f(\mathbf{x}) - E(f)\big)^2 \mu(\mathbf{x}) |d^k\mathbf{x}|.$$

Page 424 In Equation 4.2.25, the $-t^2$ in the exponent should be $-x^2$:

$$\mu(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}.$$

Page 426 In the margin note about the error function, the 2π on the left should be $\sqrt{2\pi}$:

$$\frac{1}{\sqrt{2\pi}} \int_0^a e^{-\frac{t^2}{2}}\, dt = \frac{1}{2} \operatorname{erf}\left(\frac{a}{\sqrt{2}}\right).$$

Page 428 Exercise 4.2.5: The sample space is all of $\mathbb{R}$. Part (a) should have (x) at the end:

$$\mu(x) = \frac{1}{2a} \chi_{[-a,a]}(x)$$

Part (b) should be with the chapter review exercises, as it uses material from Section 4.11.

Page 429 Caption to Figure 4.3.2, last sentence: “region is black”, not “region of is black”.

Page 431 We are not consistent in our use of notation for graphs. In Definition 3.1.1 and on this page we use $\Gamma(f)$, but on page 433 we use Γ_f and on page 778 we use $\operatorname{gr}(f)$.

Page 436 First line of second paragraph: "in Definition measuredef" should be "in Definition 4.4.1."

Page 436 In Definition 4.4.1 (and in other definitions in the text), "if and only if" is not necessary. Mathematical definitions (unlike definitions in ordinary language) are always unambiguous. However, there are other ways to define measure 0; if one used a different definition, the statement of Definition 4.4.1 would still be true, but it would be a proposition, requiring proof, and the "if and only if" would be needed.

Page 437 In the last line of the proof of Theorem 4.4.3, the second equation should be

$$\sum_{i,j} \operatorname{vol} B_{i,j} \leq \epsilon$$

(not $\operatorname{vol} X_1 \cup X_2 \cup \cdots \leq \epsilon$).

Page 438 p.438 Line 10: Example 4.3.3, not 4.4.2:
... unlike the function of Example 4.3.3, which, as far as we know, is only a pathological example, devised to test the limits of mathematical statements.

Page 440 Second paragraph of the proof of Lemma 4.4.6: $|\mathbf{x}_j - \mathbf{y}_j|$, not $|f(\mathbf{x}_j) - f(\mathbf{y}_j)|$:

"Since $|\mathbf{x}_j - \mathbf{y}_j| \to 0$ as $j \to \infty$, the subsequence $\mathbf{y}_{j_k}$ also converges to $\mathbf{p}$."

The next paragraph would perhaps be clearer if the first sentence were:

"The function f is certainly not continuous at $\mathbf{p}$, so $\mathbf{p}$ has to be in a particular box, which we will call B_p."

Page 441 We should perhaps have reminded readers that $\exists$ means "there exist." The symbol was used in Section 0.2.

Page 454 Exercise 4.5.17, part (a): "Let $M_r(\mathbf{x})$ be the rth smallest ... ", not "Let $M_r(\mathbf{x})$ be the rth largest ... ".

Page 459 In Equation 4.6.14, the sum on the right should start at $i = 1$, not $i = -k$:

$$\int_{-1}^{1} f(x)\,dx \approx \sum_{i=1}^{k} w_i\big(f(x_i) + f(-x_i)\big),$$

Page 465 Exercise 4.6.2: for $k = 1$, we meant the initial conditions to be $x_1 = .7$ and $x_2 = .5$ (not $x_1 = 17$ and $x_2 = .57$).

Page 467 Definition 4.7.2: We should have specified a *bounded* subset and a *finite* collection:

Definition 4.7.2 (A paving of $X \subset \mathbb{R}^n$). A paving of a bounded subset $X \subset \mathbb{R}^n$ is a finite collection $\mathcal{P}$ of subsets $P \subset X$ such that

$$\cup_{P \in \mathcal{P}} P = X, \text{ and } \operatorname{vol}_n(P_1 \cap P_2) = 0 \text{ (when } P_1, P_2 \in \mathcal{P} \text{ and } P_1 \neq P_2).$$

Page 468 In Definition 4.7.4 we use "diam" for "diameter," but we don't define it until page 487, just after Equation 4.9.9.

Page 468 At present, Theorem 4.7.5 requires f to be integrable. In a future edition, we will change Theorem 4.7.5 to something like

Theorem 4.7.5 (Integrals using arbitrary pavings) . *Let $X \subset \mathbb{R}^n$ be a bounded subset, and $\mathcal{P}_N$ be a nested partition of X. Suppose the boundary ∂X satisfies $\mathrm{vol}_n(\partial X) = 0$. Then $f : \mathbb{R}^n \to \mathbb{R}$ is integrable if and only if the upper and lower limits using the nested partition are equal:*

$$\lim_{N\to\infty} U_{\mathcal{P}_N}(f) = \lim_{N\to\infty} L_{\mathcal{P}_N}(f). \tag{4.7.4}$$

In that case, they are both equal to

$$\int_X f(\mathbf{x})\,|d^n\mathbf{x}|. \tag{4.7.5}$$

This will solve some problems with the current proof of Theorem 4.9.1.

Page 472 Last margin note: Definition 2.1.11 does not exist. Column operations are defined by replacing the word "row" in Definition 2.1.1 of row operations by the word "column".

Page 475 The last margin note should be on page 476.

Page 478 Equation 4.8.36: Note that when we write this permutation as $(2, 3, 1)$ we are simply dropping the left-hand side, which carries no information.

Conflicting "shorthand" notation for permutations exist. As we describe it, the notation $(3, 1, 2)$ means that the first entry goes to third place, the second goes to first, and the third goes to second. But $(3, 1, 2)$ is often interpreted as the cyclical permutation "third goes to first, which goes to second, which goes back to third": $3 \to 1 \to 2 \to 3 \ldots$.

In this cyclical notation, the permutation $\begin{bmatrix}1\\2\\1\end{bmatrix} \mapsto \begin{bmatrix}3\\2\\1\end{bmatrix}$, which leaves the second entry unchanged, would be written $(3, 1)$, i.e., $3 \to 1 \to 3$. The permutation that we would write $(3, 5, 1, 4, 2)$ would be written $(1, 3)(2, 5)$, or possible $(13)(25)$.

Page 478 Definition 4.8.15: We regret not having stated explicitly that $\mathrm{sgn}(\sigma \circ \tau) = \mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\tau$:

$$\mathrm{sgn}(\sigma \circ \tau) = \det M_{\sigma\circ\tau} = \det(M_\sigma M_\tau) = \det M_\sigma \det M_\tau = \mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\tau.$$

It was to get this equation easily that we defined the signature as we did, in terms of the determinant, which we had already defined in terms of its properties. The standard approach is to define the determinant in terms of the signature (turning Theorem 4.8.17 into a definition). This makes it excruciating to prove that $\mathrm{sgn}(\sigma\circ\tau) = \mathrm{sgn}\,\sigma\,\mathrm{sgn}\,\tau$, in order to get $\det A \det B = \det(AB)$. Of course, in mathematics, when you remove a difficulty in one place, it typically springs up someplace else; with our definition of the determinant, proving existence was not easy.

Page 480 In two places in the first line after Equation 4.8.44, $\operatorname{sgn}(\sigma)$ should be $\operatorname{sgn}(\sigma')$: "and the result follows from $\operatorname{sgn}(\tau^{-1} \circ \sigma') = \operatorname{sgn}(\tau^{-1})\big(\operatorname{sgn}(\sigma')\big) = -\operatorname{sgn}(\sigma')$, since ... "

Page 481 First line after Definition 4.8.19: one too many "is."

Page 484 Hint for Exercise 4.8.7: This hint is not actually used in the solution. Using the hint, one could write the following for part (a):

$$\det |\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{0}}, \dots, \vec{\mathbf{a}}_n| = \det |\vec{\mathbf{a}}_1, \dots, 2\vec{\mathbf{0}}, \dots, \vec{\mathbf{a}}_n| = 2 \det |\vec{\mathbf{a}}_1, \dots, \vec{\mathbf{0}}, \dots, \vec{\mathbf{a}}_n|,$$

which implies that the determinant must be 0.

Page 485 Last line of first paragraph: "volume of the parallelepiped," not area.

Page 488 In the equation following Equation 4.9.12, the left-hand side should be

$$U_{T(\mathcal{D}_N)}(\chi_{T(A)}) - L_{T(\mathcal{D}_N)}(\chi_{T(A)});$$

the upper and lower sums are with respect to the nested partition $T(\mathcal{D}_N)$.

Page 494 Discussion after Proposition 4.10.3: The reference should be to Corollary 4.3.10, not to Theorem 4.3.9. (That theorem concerns integrability, not the actual integral.)

Page 496 Last margin note: The sentence "At $\varphi = -\pi/2$ and $\varphi = \pi/2$, $r = 0$" should be deleted.

Page 502 Line 5: "in this case we can solve $xy = u$", not "in this case we can solve $y = u/v$".

Page 502 Bottom margin note: we mean to write Exercise 4.10.4, not 4.5.19.

Page 503 Exercise 4.10.3: This exercise should read

"Show that in complex notation, with $z = x + iy$, the equation of the lemniscate of Figure 4.10.3 can be written $|z^2 - \frac{1}{2}| = \frac{1}{2}$. Hint: See Example 4.10.19."

The equation given in the text is the equation for a different lemniscate.

Page 508 Caption: "first good fortune," not "first good fortunate."

Page 509 End of last margin note: "except on a set of measure 0", not "except on a measure 0."

Page 509 Theorem 4.11.8 should really come before Definition 4.11.7. The proof of the theorem is not correct; the main idea is right but there is a fiddly problem with the truncations. Here is the rewritten proof:

Proof of Theorem 4.11.8. Set $h_k = f_k - g_k$, and $H_l = \sum_{k=1}^{l} h_k$. The functions H_l form a sequence of Riemann-integrable functions converging to 0 except on a set of measure 0; if in addition they all have support in $B_R(\mathbf{0})$ and $|H_l| \le R$ for all l, then H_l meets the conditions for f_k in Theorem 4.11.4, so

$$\lim_{l\to\infty} \int_{\mathbb{R}^n} H_l(\mathbf{x})\, |d^n\mathbf{x}| = 0 \quad \text{i.e.,} \quad \lim_{l\to\infty} \sum_{k=1}^{l} \int_{\mathbb{R}^n} h_k(\mathbf{x})\, |d^n\mathbf{x}| = 0, \qquad 4.11.18$$

proving the result. We will reduce the general case, where H_l is not bounded with bounded support, to this one, by appropriately truncating the H_l.

Choose $\epsilon > 0$ and choose M such that

$$\sum_{k=M+1}^{\infty} \int_{\mathbb{R}^n} |h_k(\mathbf{x})||d^n\mathbf{x}| < \epsilon, \qquad 4.11.19$$

so that for $l > M$ we have

$$\int_{\mathbb{R}^n} |H_l(\mathbf{x}) - H_M(\mathbf{x})||d^n\mathbf{x}| \leq \sum_{k=M+1}^{l} \int_{\mathbb{R}^n} |h_k(\mathbf{x})||d^n\mathbf{x}| \leq \sum_{k=M+1}^{\infty} \int_{\mathbb{R}^n} |h_k(\mathbf{x})||d^n\mathbf{x}| < \epsilon.$$

Next choose R such that $\sup |H_M(\mathbf{x})| < R/2$ and $H_M(\mathbf{x}) = 0$ when $|\mathbf{x}| \geq R$. We will define the R-truncation of H_l by the formula

$$[H_l]_R = \sup\Big(-R\chi_{B_R(\mathbf{0})}, \inf\big(R\chi_{B_R(\mathbf{0})}, H_l\big)\Big); \qquad 4.11.20$$

i.e., replace $H_l(\mathbf{x})$ by 0 if $|\mathbf{x}| > R$, by R if $H_l(\mathbf{x}) > R$, and by $-R$ if $H_l(\mathbf{x}) < -R$, as shown in Figure 4.11.1b.

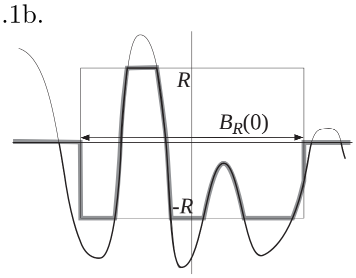


FIGURE 4.11.1B. The thin line shows the graph of f, and the dark line shows $\inf(R\chi_{B_R(\mathbf{0})}, f)$. Next we take the sup of the dark line and $-R\chi_{B_R(\mathbf{0})}$, to get the thick, light gray line representing $[f]_R$.

The $[H_l]_R$ form a sequence of Riemann-integrable functions all with support in $B_R(\mathbf{0})$ and all bounded by R, and tending to 0 except on a set of measure 0, so, by Theorem 4.11.4,

$$\underbrace{\lim_{l\to\infty} \int_{\mathbb{R}^n} [H_l]_R(\mathbf{x})\,|d^n\mathbf{x}| = 0.}_{\text{main motor of the proof}} \qquad 4.11.21$$

At this point we have done most of the work (the hard part was proving Theorem 4.11.4). But, for $l > M$, we still need to deal with the difference $H_l - [H_l]_R = (H_l - H_M) - ([H_l]_R - H_M)$. We already know that the integral of $|H_l - H_M|$ is less than ϵ, so we only need to consider the integral of $|[H_l]_R - H_M|$. Outside $B_R(\mathbf{0})$ we have $H_M = 0$ and $[H_l]_R = 0$, so

$$\int_{\mathbb{R}^n - B_R(\mathbf{0})} \big|[H_l]_R - H_M(\mathbf{x})\big||d^n\mathbf{x}| = 0. \qquad 4.11.22$$

For the integral of $|[H_l]_R - H_M|$ inside $B_R(\mathbf{0})$, first find N such that $U_N(|H_l - H_M|) < \epsilon$. Then consider the union A of the cubes $C \in \mathcal{D}_N(\mathbb{R}^n)$ that intersect

$B_R(\mathbf{0})$ and where $M_C(|H_l - H_M|) > R/2$. These have total volume at most $2\epsilon/R$, as shown by the following computation:

$$\epsilon > U_N|H_l - H_M| = \sum_{C\in\mathcal{D}_N(\mathbb{R}^n)} M_C|H_l - H_M| \operatorname{vol}_n C \geq \sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n)\\ C\subset A}} M_C|H_l - H_M| \operatorname{vol}_n C$$

$$\geq \sum_{\substack{C\in\mathcal{D}_N(\mathbb{R}^n)\\ C\subset A}} (R/2) \operatorname{vol}_n C = (R/2) \operatorname{vol}_n A.$$

Let B be the union of the cubes $C \in \mathcal{D}_N(\mathbb{R}^n)$ that intersect $B_R(\mathbf{0})$ and such that $M_C(|H_l - H_M|) \leq R/2$; on these, $|H_l| \leq |H_l - H_M| + |H_M| \leq R/2 + R/2 = R$, so $[H_l]_R = H_l$. Thus

$$\int_{B_R(\mathbf{0})} \Big|[H_l]_R(\mathbf{x}) - H_M(\mathbf{x})\Big| |d^n\mathbf{x}| \qquad 4.11.23$$

$$= \int_A \Big|[H_l]_R(\mathbf{x}) - H_M(\mathbf{x})\Big| |d^n\mathbf{x}| + \int_B \Big|[H_l]_R(\mathbf{x}) - H_M(\mathbf{x})\Big| |d^n\mathbf{x}|$$

$$\leq \frac{3R}{2} \operatorname{vol}_n(A) + \int_B \Big|H_l(\mathbf{x}) - H_M(\mathbf{x})\Big| |d^n\mathbf{x}| \leq 3\epsilon + \epsilon = 4\epsilon. \quad \square$$

Page 510 Equation 4.11.19: We meant to write the sums with $k = m + 1$, not $k = m$. (But it's correct as stated; the sums starting with $k = m$ are at least as big as the sums starting with $k = m + 1$, so either way we can go from the third to the fourth lines of Equation 4.11.21.)

A somewhat more serious issue is that if $[f_k]_R = f_k$ and $[g_k]_r = g_k$, this does not imply $[f_k - g_k]_R = f_k - g_k$. The simplest way to fix this seems to be to change Equation 4.11.19, stating explicitly that we are choosing R big enough so that:

$$\sum_{k=1}^m f_k = \sum_{k=1}^m [f_k]_R, \qquad \sum_{k=1}^m g_k = \sum_{k=1}^m [g_k]_R, \qquad \sum_{k=1}^m f_k - g_k = \sum_{k=1}^m [f_k - g_k]_R. \qquad 4.11.19$$

The left side of Equation 4.11.22 should be an absolute value:

$$\left| \int_{\mathbb{R}^n} \sum_{k=1}^p [f_k - g_k]_{2R}(\mathbf{x}) \, |d^n\mathbf{x}| \right| < \epsilon. \qquad 4.11.22$$

We perhaps should have said that Equation 4.11.22 uses the dominated convergence theorem. We have

$$\lim_{p\to\infty} \int_{\mathbb{R}^n} \sum_{k=1}^p [f_k - g_k]_{2R}(\mathbf{x}) |d^n\mathbf{x}| \underbrace{=}_{\text{dom. converg.}} \int_{\mathbb{R}^n} \underbrace{\Big(\lim_{p\to\infty} \sum_{k=1}^p [f_k - g_k]_{2R}(\mathbf{x}) \Big)}_{0 \text{ by hypothesis}} |d^n\mathbf{x}| = 0.$$

Page 511 Line after Equation 4.11.27: "at one point," not "at one points."

Page 514 The statement in the margin that "the union of sets of measure 0 has measure 0" is incorrect. It should be "the union of finitely many (or countably many) sets of measure 0 has measure 0."

The proof of Proposition 4.11.14 is not correct. It should be as follows:

Proof. Suppose $f = \sum_{k=1}^{\infty} f_k$ and $g = \sum_{k=1}^{\infty} g_k$, with all f_k, g_k R-integrable, and

$$\sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |f_k(x)|\, dx < \infty, \quad \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} |g_k(x)|\, dx < \infty. \qquad 4.11.44$$

Define $F_m = \sum_{k=1}^{m} f_k$ and $G_m = \sum_{k=1}^{m} g_k$, $H_m = \sup\{F_m, G_m\}$.

The problem is that the hypothesis $f \underset{L}{\leq} g$ does not imply that for m sufficiently large we have $F_m \underset{L}{\leq} G_m$: the inequality might go the other way on smaller and smaller sets. The solution will be to find new R-integrable functions h_k such that $g = \sum_{k=1}^{\infty} h_k$, and such that if we set $H_m = \sum_{k=1}^{m} h_k$, then indeed $F_m \underset{L}{\leq} H_m$ for all m.

Define $H_m = \sup\{F_m, G_m\}$, and $h_m = H_m - H_{m-1}$ (where we set $H_0 = 0$). finally $h_m = H_m - H_{m-1}$ (where we set $H_0 = 0$). Then certainly $F_m \leq H_m$, and

$$\sum_{m=1}^{\infty} h_m(\mathbf{x}) = \lim_{m\to\infty} H_m(\mathbf{x}) = g(\mathbf{x}). \qquad 4.11.45$$

Moreover,

$$|h_m(\mathbf{x})| = |H_m(\mathbf{x}) - H_{m-1}(\mathbf{x})| \leq \sup\{|f_m(\mathbf{x})|, |g_m(\mathbf{x})|\} \leq |f_m(\mathbf{x})| + |g_m(\mathbf{x})|.$$

We see the first inequality as follows. If the sup defining H is given by F (resp. G) for both m and $m-1$, clearly $h_m(\mathbf{x}) = f_m(\mathbf{x})$ (resp. $h_m(\mathbf{x}) = g_m(\mathbf{x})$). If it is given by F for m and by G for $m-1$, then

$$|h_m(\mathbf{x})| = |F_m(\mathbf{x}) - G_{m-1}(\mathbf{x})| \leq |F_m(\mathbf{x}) - F_{m-1}(\mathbf{x})| = |f_m(\mathbf{x})| \qquad 4.11.46$$

and similarly in the fourth case. Thus

$$\sum_{m=1}^{\infty} \int_{\mathbb{R}^n} |h_m(\mathbf{x})||d^n\mathbf{x}| \leq \sum_{m=1}^{\infty} \int_{\mathbb{R}^n} |f_m(\mathbf{x})||d^n\mathbf{x}| + \sum_{m=1}^{\infty} \int_{\mathbb{R}^n} |g_m(\mathbf{x})||d^n\mathbf{x}| < \infty. \qquad 4.11.47$$

Finally

$$\int_{\mathbb{R}^n} f(\mathbf{x})|d^n\mathbf{x}| = \sum_{k=1}^{\infty} \int_{\mathbb{R}^n} f_k(\mathbf{x})|d^n\mathbf{x}| = \lim_{m\to\infty} \int_{\mathbb{R}^n} F_m(\mathbf{x})|d^n\mathbf{x}|$$
$$\leq \lim_{m\to\infty} \int_{\mathbb{R}^n} H_m(\mathbf{x})|d^n\mathbf{x}| = \int_{\mathbb{R}^n} g(\mathbf{x})|d^n\mathbf{x}|. \qquad \square$$

Page 514 Proposition 4.11.15: a and b are constants.

Page 516 Theorems 4.11.19 and 4.11.20 are proved in Appendix A.21.

Page 518 In the last line in the margin, the third integral concerns f_2, not f_1:

$$\int f(x)dx = \int f_1(x)\, dx + i \int f_2(x)\, dx.$$

Page 520 Equation 4.11.73: An integral is missing on the second line. The equation should be

$$\begin{aligned}\lim_{h\to 0}\frac{\mathcal{L}f(s+h)-\mathcal{L}f(s)}{h} &= \lim_{h\to 0}\int_0^\infty \frac{e^{-(s+h)t}-e^{-st}}{h}f(t)\,dt\\ &= \lim_{h\to 0}\int_0^\infty f(t)e^{-st}\frac{e^{-ht}-1}{h}\,dt\end{aligned}$$

Page 521 Parts (b) of Exercises 4.11.6 and 4.11.7 are too difficult and should be deleted.

Page 522 Margin note: "not absolutely convergent," not "not absolutely convergence."

Page 526 Exercise 4.27: a sum was omitted from the definition of f. It should be

$$f(x)=\sum_{k=1}^{\infty}\frac{1}{2^k}\frac{1}{\sqrt{|x-a_k|}}.$$

Chapter 5

Page 530 Definition 5.1.3: How do we know that $\det(T^\top T)\geq 0$, so that $\sqrt{\det(T^\top T)}$ makes sense? Here is one justification:

Note that

$$\big((T^\top T)\vec{\mathbf{v}}\big)\cdot\vec{\mathbf{v}}=(T^\top T\vec{\mathbf{v}})^\top\vec{\mathbf{v}}=T\vec{\mathbf{v}}\cdot T\vec{\mathbf{v}}>0.$$

Denote by A the $k\times k$ matrix $T^\top T$ and let I be the $k\times k$ identity matrix, set $0\leq t\leq 1$, and consider the matrix $\big(tA+(1-t)I$, which we can think of as A (when $t=1$) being transformed to I (when $t=0$). Now, for $\vec{\mathbf{v}}\neq\mathbf{0}$, we have

$$\big(tA+(1-t)I\big)\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}=t\underbrace{A\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}}_{>0}+\underbrace{(1-t)}_{\geq 0}\underbrace{\vec{\mathbf{v}}\cdot\vec{\mathbf{v}}}_{>0}>0.$$

This implies that, for $0\leq t\leq 1$, $\ker\big(tA+(1-t)I=\mathbf{0}$ and thus that $\det\big(tA+(1-t)I$ is never 0 when $0\leq t\leq 1$. Since when $t=0$, $\det\big(tA+(1-t)I=1$, and when $t=1$, $\det\big(tA+(1-t)I=\det A$, it follows that $\det A>0$.

Page 531 The hint for Exercise 5.1.3 is not used in the solution given in the solution manual; in addition, it neglects to define T:

$$T=[\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k].$$

Here is a solution using the hint:

Set $T=[\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k]$. Since the vectors $\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k$ are linearly dependent, $\operatorname{rank} T<k$. Further, $\operatorname{Img} T^\top T\subset\operatorname{Img} T^\top$, so

$$\operatorname{rank} T^\top T\leq\operatorname{rank} T^\top \underbrace{=}_{\text{Prop. 2.5.12}} \operatorname{rank} T<k.$$

Since $T^\top T$ is a $k\times k$ matrix with rank $<k$, it is not invertible, hence its determinant is 0, so

$$\operatorname{vol}_k P(\vec{\mathbf{v}}_1,\dots,\vec{\mathbf{v}}_k)=\sqrt{\det T^\top T}=0.$$

Page 534 Equation 5.2.4: a_1 should be a_i in two places, and the "for $a_1, a_2, a_3, \dots$ should be omitted:

$$U = \bigcup_{i=1}^{\infty} \left(a_i - \frac{1}{2^{N+i}},\ a_i + \frac{1}{2^{N+i}} \right). \qquad 5.2.4$$

The next sentence should say "This is an open subset of $\mathbb{R}$ ... ," not "This is an open subset of $[0,1]$ ... ."

In Equation 5.2.5, the sum should start at $n = 1$ not $= 1$. On the righthand sides of Equations 5.2.5 and 5.2.6, the denominator should be 2^{N-1}, not 2^{N-2}.

Page 537 Middle margin note: z-axis, not x-axis, in "you get the equation of the surface obtained by rotating the original curve around the x-axis".

Page 539 In Figure 5.2.4, the top line in the rectangle at right should be darker.

Page 539 In Theorem 5.2.10 we used the word *diffeomorphism* without defining it. A diffeomorphism is a differentiable mapping with differentiable inverse.

Page 541 Three lines after Equation 5.3.2: "sum them," not "summ them."

Page 541 Definition 5.3.1: This definition is not wrong, but it is unfortunate that we restricted ourselves to this special case instead of defining the integral of a function over a manifold. In subsequent editions, we will replace this definition by something like

Definition 5.3.1 (Integral with respect to volume, over a manifold). Let $M \subset \mathbb{R}^n$ be a smooth k-dimensional manifold, U a pavable subset of $\mathbb{R}^k$, and $\gamma : U \to M$ a parametrization according to Definition 5.2.3. Let $f : M \to \mathbb{R}$ be a function. Then f is integrable over M with respect to volume if the last integral below exists, and then the integral is

$$\begin{aligned} \int_M f(\mathbf{x})|d^k\mathbf{x}| &= \int_{\gamma(U)} f(\mathbf{x})|d^k\mathbf{x}| \\ &= \int_U f\big(\gamma(\mathbf{u})\big)\Big(|d^k\mathbf{x}|\,\big(P_{\gamma(\mathbf{u})}\big(\overrightarrow{D_1}\gamma(\mathbf{u}), \dots, \overrightarrow{D_k}\gamma(\mathbf{u})\big)\big)\Big)|d^k\mathbf{u}| \\ &= \int_U f\big(\gamma(\mathbf{u})\big)\sqrt{\det([\mathbf{D}\gamma(\mathbf{u})]^\top[\mathbf{D}\gamma(\mathbf{u})])}\ |d^k\mathbf{u}|. \end{aligned} \qquad 5.3.3$$

Such an integral is sometimes referred to as the integral of a density, as opposed to the integral of a differential form.

If $f = 1$, the integral above gives the volume of M.

A corresponding change would then need to be made to Proposition 5.3.2 and its proof.

In several examples and exercises we actually use the above definition of "integral of a function with respect to volume."

Page 545 Line 2, plural, not singular: "the intersection of the surfaces of equations".

Equation 5.3.26: the second line should end with $d\theta$. Equation 5.3.27: This equation should not have a $d\theta$ at the end. It should have a period.

Page 549 In three places, D_2f should be D_3f: the last line of Equation 5.3.45 should be

$$1 + (D_1f)^2 + (D_2f)^2 + (D_3f)^2;$$

In the second line of Equation 5.3.45, three closing parentheses aren't opened. The line should be

$$= \det \begin{bmatrix} 1+(D_1f)^2 & (D_1f)(D_2f) & (D_1f)(D_3f) \\ (D_1f)(D_2f) & 1+(D_2f)^2 & (D_2f)(D_3f) \\ (D_1f)(D_3f) & (D_2f)(D_3f) & 1+(D_3f)^2 \end{bmatrix}$$

Equation 5.3.46 should be

$$\int_U \sqrt{1 + (D_1f)^2 + (D_2f)^2 + (D_3f)^2}|d^3\mathbf{x}|,$$

and the left-hand side of the first line of Equation 5.3.48 should be

$$\int_{B_\mathbf{0}(R)} \sqrt{1 + (D_1f)^2 + (D_2f)^2 + (D_3f)^2}|d^3\mathbf{x}|.$$

Page 550 The caption to Table 5.3.3 would perhaps be clearer as follows: Computing the volume of the n-dimensional unit ball in $\mathbb{R}^n$, for $n = 1, \dots, 5$, and for the n-dimensional unit sphere in $\mathbb{R}^{n+1}$, for $n = 0, 1 \dots, 5$. (The 0-dimensional sphere in $\mathbb{R}$ consists of the two points -1 and 1.)

Page 551 Exercise 5.3.2: "Use the result of Exercise 5.3.1 (a)", not "use Equation 5.3.1 … ".

Page 552 first margin note: the earth's circumference, not diameter!

Exercise 5.3.12: The total curvature of a curve C is $\int_C \kappa\,|d^1\mathbf{x}|$.

Page 556 Exercise 5.6: Some subscripts got forgotten, and one superscript is wrong. It should be:

(a) Show that $w'_{n+1}(r) = v_n(r)$.

(b) Show that $v_n(r) = r^n v_n(1)$.

(c) Derive Equation 5.3.49, using $w_{n+1}(1) = \int_0^1 w'_{n+1}(r)\,dr$.

Chapter 6

Page 561 Definition 6.1.3. "An elementary k-form", not "A elementary k-form".

Page 562 The right side of Equation 6.1.14 should be

$$\sum_{i=1}^{k-1} a_i \varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_{k-1}, \vec{\mathbf{v}}_i).$$

The first term is $a_1\phi(\vec{\mathbf{v}}_1, ..., \vec{\mathbf{v}}_{k-1}, \vec{\mathbf{v}}_1)$, the second is $a_2\phi(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2, ..., \vec{\mathbf{v}}_{k-1}, \vec{\mathbf{v}}_2)$, and so on.

Page 563 Clarification for Example 6.1.8:

The function $W_{\vec{\mathbf{v}}}(\vec{\mathbf{w}}) = \vec{\mathbf{v}} \cdot \vec{\mathbf{w}}$ is a 1-form on $\mathbb{R}^n$ because it is a function of one vector and it is linear as a function of $\vec{\mathbf{w}}$. The requirement that it be antisymmetric is automatically satisfied, since it is a function of only one vector.

Page 564 Equation 6.1.23 should be

$$dx_{i_1} \wedge \cdots \wedge dx_{i_k}(\vec{\mathbf{e}}_{j_1}, \ldots \vec{\mathbf{e}}_{j_k}). \qquad 6.1.23$$

Equation 6.1.24 should be

$$dx_{j_1} \wedge \cdots \wedge dx_{j_k}(\vec{\mathbf{e}}_{j_1}, \ldots, \vec{\mathbf{e}}_{j_k}) = 1. \qquad 6.1.24$$

Page 568 Not an error, but in subsequent editions we plan to add the following to the first margin note:

If V is k-dimensional, a nonzero element of $A^k(V)$ will correspond, via $\Phi_{\{\underline{\mathbf{b}}\}}$ as in Equation 6.1.30, to a nonzero multiple of $\det \in A^k(\mathbb{R}^k)$. In particular, a nonzero element of $A^k(V)$ evaluated on k linearly independent vectors always returns a nonzero number.

Page 569 The last margin note refers to nonexistent parts a) and b) of Definition 6.1.1. That sentence should read

The wedge product $\varphi \wedge \omega$ satisfies the requirements of Definition 6.1.1 for a form (multilinearity and antisymmetry).

Page 570 Discussion after Definition 6.1.22:

We will assume that these functions are of class at least C^2: we will need C^1 to define the exterior derivative and C^2 for Theorem 6.7.7 to be true.

Page 571 Exercise 6.1.2 (a): $dx_3 \wedge dx_2 \wedge x_4$ should be $dx_3 \wedge dx_2 \wedge dx_4$.

Page 580 Caption to Figure 6.3.1: "we choose a tangent vector field", not "we choose tangent vector field".

After Definition 6.3.1, add

If (M, ω) is a manifold oriented by the form ω, then $-(M, \omega)$ will refer to M with the opposite orientation. It follows that $-(M, \omega) = (M, -\omega)$.

Page 581 Proposition 6.3.5: As written, this proposition assumes that an appropriate normal vector field can be chosen. Of course, that is not always the case, as is clear from considering the Moebius strip. The proposition should read

Proposition 6.3.5 (Orienting a surface in $\mathbb{R}^3$). *Let $S \subset \mathbb{R}^3$ be a smooth surface. In this case $T_{\mathbf{x}}S$ is two-dimensional, and an element of the line $A^2(T_{\mathbf{x}}S)$ is a 2-form. Suppose there exists a normal vector field $\vec{\mathbf{n}}$, as shown in Figure 6.3.2: for each $\mathbf{x} \in S$ we can choose a nonzero vector $\vec{\mathbf{n}}(\mathbf{x}) \in T_{\mathbf{x}}S^\perp$, such that $\vec{\mathbf{n}}(\mathbf{x})$ varies continuously with $\mathbf{x}$. Then S can be oriented by the 2-form field $\omega_{\mathbf{x}} \in A^2(T_{\mathbf{x}}S)$ given by*

$$\omega_{\mathbf{x}}(\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2) = \det[\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2], \quad \text{where } \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2 \in T_{\mathbf{x}}S. \qquad 6.3.4$$

In the proof, we should write "$\omega_{\mathbf{x}}$ is not the zero element of $A^2(T_{\mathbf{x}}S)$," not "$\omega_{\mathbf{x}}$ is not the 0-form":

Proof. The 2-form $\omega_{\mathbf{x}}$ is not the zero element of $A^2(T_{\mathbf{x}}S)$, since if $\vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are linearly independent and are in $T_{\mathbf{x}}S$, then $\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2$ are linearly independent, with nonzero determinant; $\omega_{\mathbf{x}}$ varies continuously because $\det[\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_1, \vec{\mathbf{v}}_2]$ is a polynomial, and (Corollary 1.5.30) polynomial functions are continuous. $\square$

Page 582 Proposition 6.3.8: We should have said "Suppose there exists a normal vector field $\vec{\mathbf{n}}$", not "Choose a normal vector field $\vec{\mathbf{n}}$". If no normal vector field $\vec{\mathbf{n}}$ exists, then the manifold is not orientable.

Page 583 In the second line of proof of Proposition 6.3.9, an end parenthesis is missing: $A^0(\{\vec{\mathbf{0}}\}) = \mathbb{R}$, not $A^0(\{\vec{\mathbf{0}}\} = \mathbb{R}$.

Page 584 In Equation 6.3.9, the second equality is incorrect; the second determinant is opposite the first. The discussion should read:

... so we are looking for either

$$\omega_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \det \begin{bmatrix} y & 0 & v_1 & w_1 \\ x & 2y & v_2 & w_2 \\ w & 2z & v_3 & w_3 \\ z & 0 & v_4 & w_4 \end{bmatrix} \quad \text{or} \quad \omega'_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = \det \begin{bmatrix} 0 & y & v_1 & w_1 \\ 2y & x & v_2 & w_2 \\ 2z & w & v_3 & w_3 \\ 0 & z & v_4 & w_4 \end{bmatrix}. \qquad 6.3.9$$

These 2-forms are nonzero elements of $A^2(T_{\mathbf{x}}S)$, i.e., $\omega_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}}) = -\omega'_{\mathbf{x}}(\vec{\mathbf{v}}, \vec{\mathbf{w}}) \neq 0$ if $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in T_{\mathbf{x}}S$ are linearly independent. The first gives

$$\begin{aligned} \omega_{\mathbf{x}} = -2z^2\, dx \wedge dy + 2yz\, dx \wedge dz + (2xz - 2yw)\, dx \wedge dw \\ + 2y^2\, dz \wedge dw - 2zy\, dy \wedge dw. \end{aligned} \qquad 6.3.10$$

Page 584 The footnote is not well written. It should be replaced by

"A nonzero k-form on a k-dimensional vector space returns 0 when evaluated on k vectors if and only if the vectors are linearly dependent."

Page 589 Part (c) of Exercise 6.3.12: The notation is inconsistent. We will change $\mathbf{v}_1$ to $\mathbf{v}$ and $\mathbf{v}_2$ to $\mathbf{w}$:

(c) Show that given any two linearly independent vectors $\mathbf{u}_1, \mathbf{u}_2$ in $\mathbb{R}^n$, $n > 2$, there exist maps $\mathbf{v}, \mathbf{w} : [0,1] \to \mathbb{R}^n$ such that

$$\mathbf{v}(0) = \mathbf{u}_1,\ \mathbf{v}(1) = \mathbf{u}_2,\ \mathbf{w}(0) = \mathbf{u}_2,\ \mathbf{w}(1) = \mathbf{u}_1,$$

and for each t, $\mathbf{v}(t)$ and $\mathbf{w}(t)$ are linearly independent.

Page 590 Exercise 6.3.17, part (b): The curve C should be smooth.

Page 591 Third margin note: Definition 6.4.2, not 6.4.1.

Page 592 Footnote: "It is never the 0-form" should be "it is never the zero element of $A^2(T_{\mathbf{x}}S)$."

Page 592 We forgot to put a $\triangle$ to mark the end of Example 6.4.3.

Page 595 Second line in margin: pullback of ω, not pullback of φ.

Margin note half-way down the page: Equation 6.4.20, not 6.4.19.

Page 596 First margin note, third line: there is an extra colon.

Page 602 The solution to Exercise 6.4.6 uses the formula

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Justifying this formula uses three statements taught in one-variable calculus and the fact (Proposition 1.5.34) that absolute convergence implies convergence. The three statements are the expression of $\sin t$, $\cos t$, and e^t, for t real, in terms of power series:

$$\begin{aligned} \sin t &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \\ \cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \\ e^t &= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots . \end{aligned} \tag{1}$$

First, let us show that for a complex number z, we can define e^z by the power series

$$e^z = 1 + z + \frac{z^2}{2!} + \cdots .$$

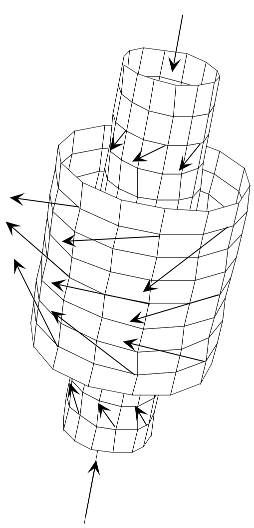


FIGURE 6.5.4.
Corrected figure

We know it is true in the special case where z is real. We need to check that the series converges. The series $1+|z|+\left|\frac{z^2}{2!}\right|+\cdots$ converges, since (by Equation (1): $|z|$ is a real number)

$$\sum_{k=0}^{\infty}\left|\frac{z^k}{k!}\right| = \sum_{k=0}^{\infty}\frac{|z|^k}{k!} = e^{|z|}$$

converges. So Proposition 1.5.34 says that $\displaystyle\sum_{k=0}^{\infty}\frac{z^k}{k!}$ converges.

Now write

$$\begin{aligned} \cos t + i\sin t &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) + i\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} -\right) \\ &= \left(1 + \frac{(it)^2}{2!} + \frac{(it)^4}{4!} + \frac{(it)^6}{6!} + \cdots\right) + \left(it + \frac{(it)^3}{3!} + \frac{(it)^5}{5!} +\right) \\ &= 1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \frac{(it)^4}{4!} + \cdots = e^{it}. \end{aligned}$$

Page 603 Caption for Figure 6.5.2: In two places (the first line and immediately after the displayed equation), $x\,dx + y\,dx$ should be $x\,dx + y\,dy$.

Page 603 The sentence "the requirement of antisymmetry then says that $f(-P_{\mathbf{x}}) = -f(\mathbf{x})$" should be deleted.

Page 605 Figure 6.5.4: the vector field should turn clockwise, as shown in the margin.

Page 606 Line 4: clockwise, not counter-clockwise.

Page 607 There were two mistakes in Example 6.5.6 The tangent vector field is $\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$, not $\begin{bmatrix} -\sin t \\ \cos t \\ 0 \end{bmatrix}$, and $\vec{\gamma}'(t)$ is $\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$. Thus the first half of the example should read:

What is the work of the vector field $\vec{F}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{bmatrix} y \\ -x \\ 0 \end{bmatrix}$ over the helix oriented by the tangent vector field $\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}$, and parametrized by $\gamma(t) = \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$, for $0 < t < 4\pi$?

The parametrization preserves orientation, since

$$\omega\big(\vec{\gamma}'(t)\big) = \underbrace{\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}}_{\vec{\mathbf{t}}(t)} \cdot \underbrace{\begin{bmatrix} -\sin t \\ \cos t \\ 1 \end{bmatrix}}_{\vec{\gamma}'(t)} = 2 > 0. \qquad 6.5.13$$

Page 607 Immediately before Equation 6.5.15: "orientation-preserving", not "orientation-preseving".

Page 609 Last margin note: the signs are reversed in the matrix; it should be $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

Page 616 Definition 6.6.2, part (2): $\left[\mathbf{D}\begin{pmatrix} \mathbf{f} \\ g \end{pmatrix}\big(\mathbf{x}\big)\right]$, not $\left[\mathbf{D}\begin{matrix} \mathbf{f} \\ g \end{matrix}\big(\mathbf{x}\big)\right]$

Page 618 Immediately before Example 6.6.6 we have added

An *oriented* piece-with-boundary of a manifold is a piece of an oriented manifold: the piece inherits the orientation of the manifold. Given $X \subset (M, \omega)$, we write $-X$ to denote X as a subset of $-M$.

Page 619 Caption to Figure 6.6.7, last sentence:
"However, the two-dimensional...", not "However, that the two-dimensional...".

Equation 6.6.5 has a misplaced end parenthesis; the first equation should be

$$g(\mathbf{y}) = (\mathbf{y} - \mathbf{0}) \cdot \vec{\mathbf{w}}_i = 0.$$

Page 619 Example 6.6.7: We changed the second paragraph to read

Let $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ be linearly independent vectors in $\mathbb{R}^n$. We will show that the parallelogram $P_{\mathbf{0}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is a piece-with-boundary of the subspace $M \subset \mathbb{R}^n$ spanned by those vectors, i.e., $M = \operatorname{Sp}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$.

In doing so we removed the part about $\mathbf{f}$, which we put into the fourth paragraph:

First we will show that any point that is in a face and is not in any edge is a smooth point. Choose vectors $\vec{\mathbf{w}}_1, \dots, \vec{\mathbf{w}}_k$ in M so that $\vec{\mathbf{w}}_i$ is orthogonal to $\vec{\mathbf{v}}_1, \dots, \widehat{\vec{\mathbf{v}}_i}, \dots \vec{\mathbf{v}}_k$; change the sign if necessary, so that $\vec{\mathbf{w}}_i \cdot \vec{\mathbf{v}}_i > 0$. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be a linear transformation whose kernel is precisely M; note that $\mathbf{f}$ is necessarily surjective. Then $P_{\mathbf{0}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is defined by the equalities and inequalities

Page 622 Last line of Definition 6.6.10: $\partial_1 P$ should be $\partial_M P$.

Page 622 Notational inconsistency. We use both ω^∂ and ω_∂ for the form orienting the boundary. In future printings we will stick with ω^∂.

Page 623 Equation 6.6.16 has a superfluous end parenthesis; it should be

$$\omega^\partial_{\mathbf{x}}(\vec{\mathbf{v}}) = \det\big(\vec{\mathbf{n}}(\mathbf{x}), \vec{\mathbf{v}}_{\text{out}}, \vec{\mathbf{v}}\big).$$

Page 623 The first four lines of the new subsection now read

We saw earlier that an oriented k-parallelogram $P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is a piece-with-boundary of $\operatorname{Sp}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ when those vectors are linearly independent. Since $\operatorname{Sp}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is oriented by the order of the vectors, a k-parallelogram is an *oriented* piece-with-boundary. As such its boundary carries an orientation.

In addition, we added this as a margin note:

Recall (Proposition 6.3.9) that a 0-dimensional manifold is oriented by the choice of sign. Thus an oriented 0-parallelogram $P_{\mathbf{x}}$ is either $+P_{\mathbf{x}}$ or $-P_{\mathbf{x}}$. (Recall from the remark immediately after Definition 6.3.13 that the description of orientation in terms of direct bases does not work in the 0-dimensional case.) Since $P_{\mathbf{x}}$ is itself a manifold, its boundary is empty, which is what Proposition 6.6.15 says when $k = 0$.

Page 626 Exercise 6.6.1: The way this exercise was stated in the first printing was not optimal; it should say:

"Use Definition 5.2.1 to show that a single point in any $\mathbb{R}^n$ never has 0-dimensional volume 0."

Page 626 Exercise 6.6.5: $\vec{\nabla}$ denotes the transpose of the derivative:

$$\vec{\nabla} f(\mathbf{x}) = [\mathbf{D}f(\mathbf{x})]^\top.$$

For a function $f : \mathbb{R}^n \to \mathbb{R}$, $\vec{\nabla} f(\mathbf{x})$ is a vector whereas $[\mathbf{D}f(\mathbf{x})]$ is a line matrix.

Note that $\vec{\nabla} f(\mathbf{x}_0)$ is orthogonal at $\mathbf{x}_0$ to the manifold X of equation $f(\mathbf{x}) = 0$: since $T_{\mathbf{x}_0}X = \ker[\mathbf{D}f(\mathbf{x}_0)]$, if $\vec{\mathbf{v}} \in T_{\mathbf{x}_0}X$, then

$$\vec{\nabla} f(\mathbf{x}_0) \cdot \vec{\mathbf{v}} = [\mathbf{D}f(\mathbf{x}_0)]\vec{\mathbf{v}} = 0.$$

Page 627 Exercise 6.6.8 should say that M is oriented by $dx_1 \wedge dx_2 \wedge dx_3$. In Exercise 6.6.8, "at a point of ∂M" should be "at a point of ∂X".

Page 629 Last line of Remark 6.7.2: "in higher dimensions", not "to higher dimensions".

Page 631 In the first line of Equation 6.7.14, φ should be ψ.

Page 632 In Theorem 6.7.7, we should have said, "For any k-form φ of class C^2 ... ."

Page 633 Second margin note: Theorem 6.7.8, not A6.7.8.

Page 634 Exercise 6.7.6: "Compute the following exterior derivatives," not "Compute the exterior following derivatives."

Exercise 6.7.7: In part (b), "check the computation in (b)" should be "check the computation in (a)."

Exercise 6.7.10: "face" rather than "edge" in two places.

Page 635 Note: The formulas for the gradient and the divergence work in any $\mathbb{R}^n$, but there is no obvious generalization of the curl, other than the exterior derivative.

Page 636 The last term on the right-hand side of Equation 6.8.5 should be $D_3 f v_3$, not $D_3 v_3$.

Page 639 The geometric interpretation of the curl that is given applies equally to $\operatorname{curl} F$ and $-\operatorname{curl} F$. It should read:

The curl probe. Consider an axis, free to rotate in a bearing that you hold, and having paddles attached, as in Figure 6.8.2. If you stand this paddle wheel on a table, paddle end down, next to a clock lying flat on the table, then the wheel turns clockwise if it follows the motion of the hands of the clock. We will orient the axis of the probe up, away from the paddle. We will assume that the bearing is packed with a viscous fluid, so that its angular speed (not acceleration) is proportional to the torque exerted by the paddles. If a fluid is in constant motion with velocity vector field $\vec{F}$, then the curl of the velocity vector field at $\mathbf{x}$, $(\vec{\nabla} \times \vec{F})(\mathbf{x})$, is measured as follows:

> *Insert the paddle of the curl probe into the vector field at a point* $\mathbf{x}$ *and adjust it so that it is spinning counterclockwise the fastest. Then the curl of the vector field at* $\mathbf{x}$ *points in the direction of axis of the probe. The speed at which the probe spins is proportional to the magnitude of the curl.*

Page 640 In the margin note, curl $\vec{F}$ should be curl curl $\vec{F}$. In $\mathbb{R}^3$ the Laplacian is often denoted Δ. Note that Δ is the dot product $\nabla \cdot \nabla$:

$$\begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} \cdot \begin{bmatrix} D_1 \\ D_2 \\ D_3 \end{bmatrix} = D_1^2 + D_2^2 + D_3^2.$$

Thus Δ is sometimes denoted ∇^2.

Page 642 We omitted part (c) of Exercise 6.8.10:

(c) Compute it again, directly from the definition of the exterior derivative.

Page 642 Part (c) of Exercise 6.8.11 was not clearly stated. We mean that you should compute them directly from the definition of the exterior derivative. We strongly recommend doing at least part of part (c).

Page 650 Exercise 6.9.6: We should have specified $a, b > 0$ and we should have discussed orientation. Future editions will contain a new part (b):

(b) Show that $(x_1\, dx_2 - x_2\, dx_1) \wedge (x_3\, dx_4 - x_4\, dx_3)$ is an orientation of the surface. Does your parametrization preserve or reverse orientation?

The current parts (b), (c), and (d) will become (c), (d), and (e).

Page 651 Theorem 6.10.2: "Definition 6.6.13" should be "Definition 6.6.10".

Page 657 Exercise 6.10.8 is wrong as written; indeed, it contradicts Exercise 6.10.7. The vector fields should be $\begin{bmatrix} xy^2 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -x^2y \end{bmatrix}$.

Page 658 Exercise 6.10.15, part (b): "the surface $X_{p,q}$ of equation $z_1^p + z_2^q$ should be "the surface $X_{p,q}$ of equation $z_1^p + z_2^q = 0$."

Page 659 We should have chosen our bicycle trip at the top of the hill; then it would be clear that if a cyclist starts and ends at the same point, he or she does no work against gravity. In the absence of friction (including friction from braking) a cyclist could zoom down one hill and coast back up the next, without doing any work.

Page 661 Margin note: Equation 6.5.12, not 5.6.1.

Page 662 The function described in Theorem 6.11.5 is unique up to the addition of an arbitrary constant. Thus the function given in Equation 6.11.24 is not the only potential of the vector field; any function $\frac{xy^2}{2} + xyz + c$, where c is an arbitrary constant, is also a potential of $\vec{F}$.

Page 664 Exercise 6.1.3, part (b): "Sketch the potential" should be "sketch the electric field."

Page 665 Exercise 6.11: "for the following 1-forms on $\mathbb{R}^2$ should be "for the following 1-forms."

Page 666 Exercise 6.12: the matrix should be $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. This affects parts (a) and (b).

Page 667 In Exercise 6.18, part (b), the displayed equation should be

$$\mathrm{vol}_n\big(B_1^n(\mathbf{0})\big) = \frac{1}{n}\,\mathrm{vol}_{n-1}(S^{n-1}).$$

Appendix A

Page 670 In the first sentence after Definition A1.2, $\mathrm{Assoc}(x, y) = (x+y)+z$ should be $\mathrm{Assoc}(x, y, z) = (x + y) + z$.

The words "k-close" were omitted from Definition A1.3, which should read

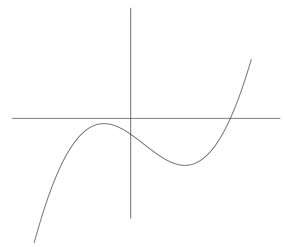
Definition A1.3 (k-close). Two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ are k-close if for each $i = 1, \dots, n$, then $\big|[x_i]_k - [y_i]_k\big| \le 10^{-k}$.

Page 671 Exercise A1.2 left out "$\mathrm{Assoc}(x, y, z) =$." The first sentence of the exercise should read

"Show that the functions $A(x, y) = x + y$, $M(x, y) = xy$, $S(x, y) = x - y$, and $\mathrm{Assoc}(x, y, z) = (x + y) + z$ are $\mathbb{D}$-continuous, and that $1/x$ is not."

Page 675 Proposition A2.4: By "exactly" we mean "if and only if." In any case, "if and only if" is more appropriate here. We tend to use "precisely" (or, more rarely, "exactly") when we mean "if and only if" but where the result is fairly obvious, which isn't the case here.

The bottom graph in Figure A2.1 is wrong; it should be:



Page 682 Restatement of Theorem 2.7.13: in the next-to-last line, it should be "has a unique solution in the closed ball $\overline{U_0}$".

Page 691 We corrected Equation 2.9.13 in Section 2.9 (page 270). Of course it should also be corrected here:

$$R_1 \;=\; R|L^{-1}|^2 \left(\sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}} - |L| \right). \qquad 2.9.13$$

Appendix A.8 The proof is not as clear as it should be as to why the root found by Newton's method is unique in all of W_0 and not just in U_0. This question is addressed by part (3) of the proof of the *inverse* function theorem, which refers to Remark A5.5 on page 688. Since we treat the implicit function theorem as a special case of the inverse function theorem, this is relevant. In any future editions we plan to put the content of Remark A5.5 in Section 2.7, perhaps immediately after the statement of the Kantorovich theorem.

Page 692 This correction was made in the text files some time ago but we forgot to include it in the errata files. The proof of Theorem 2.9.7 does not include a proof of the last statement, concerning Equation 2.9.13. Here is the missing proof:

Proving Equation 2.9.13

Suppose $|\mathbf{x} - \mathbf{x}_0| < R_1$. Then

$$|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}_0)| \;\le\; |\mathbf{x} - \mathbf{x}_0| \sup |[\mathbf{Df}(\mathbf{x})]| \;\le\; R_1 \sup |[\mathbf{Df}(\mathbf{x})]|. \qquad A7.11$$

We find a bound for $|[\mathbf{Df}(\mathbf{x})]|$:

$$|[\mathbf{Df}(\mathbf{x})] - [\mathbf{Df}(\mathbf{x}_0)]| = |[\mathbf{Df}(\mathbf{x})] - L| \underbrace{\le}_{\text{Eq. 2.9.11}} \frac{1}{2R|L^{-1}|^2} |\mathbf{x} - \mathbf{x}_0| \le \frac{R_1}{2R|L^{-1}|^2}$$

so

$$|[\mathbf{Df}(\mathbf{x})]| \le |L| + \frac{R_1}{2R|L^{-1}|^2}, \quad \text{i.e.,} \quad \sup |[\mathbf{Df}(\mathbf{x})]| = |L| + \frac{R_1}{2R|L^{-1}|^2}. \qquad A7.12$$

Therefore we want to find the largest R_1 satisfying

$$R \;\ge\; \left(|L| + \frac{R_1}{2R|L^{-1}|^2} \right) R_1. \qquad A7.13$$

The right-hand side is 0 when $R_1 = 0$ and then increases as R_1 increases, so we want the largest value of R_1 for which the inequality is an equality. Thus we want to solve the quadratic equation

$$R_1^2 + 2R|L^{-1}|^2\,|L|R_1 - 2R^2|L^{-1}|^2 = 0, \qquad \text{A7.14}$$

which gives

$$R_1 = R|L^{-1}|^2\left(-|L| + \sqrt{|L|^2 + \frac{2}{|L^{-1}|^2}}\right). \qquad \text{A7.15}$$

Page 695 Equation A8.6 is wrong. It should be

$$\mathbf{F}\begin{pmatrix}\mathbf{g}(\mathbf{y})\\ \mathbf{y}\end{pmatrix} = \mathbf{0}.$$

Page 705 Second line after Equation A11.16: it might be clearer to write "which satisfy $|Q^k_{f,\mathbf{a}}(\vec{\mathbf{h}})| \in O(|\vec{\mathbf{h}}|)$", rather than "so that $|Q^k_{f,\mathbf{a}}(\vec{\mathbf{h}})| \in O(|\vec{\mathbf{h}}|)$."

Page 707 Equation A12.3 should end with ds, not dt.

Page 723 In the next-to-last line of the paragraph beginning "Fortunately", the word "volume" should be "measure".

Page 724 Corollary A16.3 is wrong. It is correct if we replace "volume" by "measure." Seeing why the proof is correct requires the following corollary to Theorem 4.4.5:

If f and g are integrable functions on $\mathbb{R}^n$, $g \geq f$, and $\int f(\mathbf{x})|d^n\mathbf{x}| = \int g(\mathbf{x})|d^n\mathbf{x}|$, then $\{\, \mathbf{x} \mid f(\mathbf{x}) \neq g)\mathbf{x})\,\}$ has measure 0.

We propose making this into an exercise, with the hint: Show that if $g(\mathbf{x}_0) > f(\mathbf{x}_0)$ and $g-f$ is continuous at $\mathbf{x}_0$, then $\int g(\mathbf{x})|d^n\mathbf{x}| > \int f(\mathbf{x})|d^n\mathbf{x}|$. Then apply Theorem 4.4.5.

Page 726 Second line of the proof: replacing f by $\chi_X f$ uses the fact that the product of two R-integrable functions is integrable. This is proved in Corollary 4.4.8; it also follows from Theorem 4.3.1. (But the product of two L-integrable functions is not necessarily L-integrable! However, the product of an L-integrable function by a bounded L-integrable function is L-integrable; see the lemma – a somewhat weaker statement – discussed in the note for page 754.)

Page 727 In the first line, we write that every $\mathbf{x}$ is in some paving tile. It is possible that $\mathbf{x}$ may be in more than one tile. By Corollary 4.3.10, such points don't affect integals; however, the definition of $\overline{g}$ should take such points into account:

$$\overline{g}(\mathbf{x}) = \begin{cases} M_{P_{N''}(\mathbf{x})}(f) & \text{if } P_{N''}(\mathbf{x}) \cap \partial\mathcal{D}_N = \phi \text{ and } \mathbf{x} \text{ is contained in a} \\ & \quad \text{single tile} \\ -\sup|f| & \text{otherwise.} \end{cases}$$

We have also rewritten some of the rest of the page, in hopes of making it clearer:

Now we compute the upper sum $U_{\mathcal{P}_{N''}}(f)$, as follows:

$$U_{\mathcal{P}_{N''}}(f) = \sum_{P \in \mathcal{P}_{N''}} M_P(f) \operatorname{vol}_n P \qquad \text{A17.8}$$

$$= \underbrace{\sum_{\substack{P \in \mathcal{P}_{N''}, \\ P \cap \partial \mathcal{D}_N = \phi}} M_P(f) \operatorname{vol}_n P}_{\substack{\text{contribution from } P \\ \text{entirely in dyadic cubes}}} + \underbrace{\sum_{\substack{P \in \mathcal{P}_{N''}, \\ P \cap \partial \mathcal{D}_N \neq \phi}} \big(M_P(f) \operatorname{vol}_n P.}_{\substack{\text{contribution from } P \text{ that intersect} \\ \text{the boundary of dyadic cubes}}}$$

We want a statement that relates integrals computed using dyadic cubes and paving tiles. Since $\sum_{P \in \mathcal{P}} \chi_P = 1$ except on a set of volume 0,

The sum of characteristic functions is the constant function 1 except on a set of volume 0.

$$\int_{\mathbb{R}^n} \overline{g}(\mathbf{x}) |d^n \mathbf{x}| = \sum_{P \in \mathcal{P}''_N} \int_{\mathbb{R}^n} \overline{g}(\mathbf{x}) \chi_P(\mathbf{x}) |d^n \mathbf{x}|$$

$$= \sum_{\substack{P \in \mathcal{P}_{N''}, \\ P \cap \partial \mathcal{D}_N = \phi}} M_P(f) \operatorname{vol}_n P + \sum_{\substack{P \in \mathcal{P}_{N''}, \\ P \cap \partial \mathcal{D}_N \neq \phi}} (-\sup |f|) \operatorname{vol}_n P.$$

Note that we can write the last term in Equation A17.8 as

$$\sum_{\substack{P \in \mathcal{P}_{N''}, \\ P \cap \partial \mathcal{D}_N \neq \phi}} \big(M_P(f) \operatorname{vol}_n P = \sum_{\substack{P \in \mathcal{P}_{N''}, \\ P \cap \partial \mathcal{D}_N \neq \phi}} \Big(M_P(f) \overbrace{- \sup |f| + \sup |f|}^{\text{cancels out}}\Big) \operatorname{vol}_n P$$

$$= \sum_{\substack{P \in \mathcal{P}_{N''}, \\ P \cap \partial \mathcal{D}_N \neq \phi}} \big(- \sup |f|\big) \operatorname{vol}_n P + \sum_{\substack{P \in \mathcal{P}_{N''}, \\ P \cap \partial \mathcal{D}_N \neq \phi}} \big(M_P(f) + \sup |f|\big) \operatorname{vol}_n P.$$

Since $M_P(f)$ is the least upper bound over P while $\sup |f|$ is the least upper bound over $\mathbb{R}^n$, we have $M_P(f) + \sup |f| \leq 2 \sup |f|$.

So we can rewrite Equation A17.8 as

$$U_{\mathcal{P}_{N''}}(f) = \int_{\mathbb{R}^n} \overline{g} |d^n \mathbf{x}| + \sum_{\substack{P \in \mathcal{P}_{N''}, \\ P \cap \partial \mathcal{D}_N \neq \phi}} \overbrace{\big(M_P(f) + \sup |f|\big)}^{\leq 2 \sup |f| \text{ (see note in margin)}} \operatorname{vol}_n P. \qquad \text{A17.11}$$

Pages 742, 743, 745 Each page has "integrable functions" that should be "R-integrable functions":

Proposition A21.1: "Let $f_k : \mathbb{R}^n \to \mathbb{R}$ be a sequence of R-integrable functions . . . "

Corollary A21.2: "Let h_k be a sequence of R-integrable nonnegative functions on Q . . . "

Proposition A21.3: "Suppose f_k is a sequence of R-integrable functions all satisfying . . . "

(These statements come in in the course of proving Theorem 4.11.4, which is a statement about Riemann integrals.)

Page 744 The last sum in Equation A21.8 should have i, not k:

$$\sum_{i=1}^{\infty} \int h_i |d^n \mathbf{x}|.$$

Page 747 The letter A is in the wrong font in one place (A should be A):

"But this argument requires "measure 0." To apply it to the case where $A \neq 0 \ldots$ "

Page 748 Equation A21.26: In the first line, the sums should be over $C \subset Y_0$, not $C \in Y_0$. But then we also have to specify that the C are in $\mathcal{D}_{N_0}(\mathbb{R}^n)$. This gives

$$\begin{aligned} \mathrm{vol}_n(Y_0)\frac{A}{\epsilon} = \sum_{\substack{C\subset Y_0 \\ C\in\mathcal{D}_{N_0}(\mathbb{R}^n)}} \frac{A}{\epsilon}\,\mathrm{vol}_n(C) \;\leq\; \sum_{\substack{C\subset Y_0 \\ C\in\mathcal{D}_{N_0}(\mathbb{R}^n)}} M_C(h_0)\,\mathrm{vol}_n(C) \\ \leq \sum_{C\in\mathcal{D}_{N_0}} M_C(h_0)\,\mathrm{vol}_n(C) = U_{N_0}(h_0) \leq 2A, \end{aligned} \qquad \text{A21.26}$$

In Equation A21.29, h_m comes with a + sign and h_{m+1} with a − sign; it should be reversed. In the second line, the = should be <. So the equation should read

$$\begin{aligned} \int_{\mathbb{R}^n} g_{m+1}(\mathbf{x})|d^n\mathbf{x}| = \left(\int_{\mathbb{R}^n} h_{m+1}(\mathbf{x})\,|d^n\mathbf{x}| - A\right) - \left(\int_{\mathbb{R}^n} h_m(\mathbf{x})\,|d^n\mathbf{x}| - A\right) \\ \leq \frac{A}{4^{m+3}} + \frac{A}{4^{m+2}} < \frac{A}{2\cdot 4^{m+1}}. \end{aligned} \qquad \text{A21.29}$$

Page 749 Equation A21.35 : on the far right, the A in the numerator should be ϵ:

$$\mathrm{vol}_n(Y_{m+1}) \leq \epsilon\frac{2^{m+1}}{4^{m+1}} = \frac{\epsilon}{2^{m+1}}.$$

Page 750 Equation A21.39: Writing "for $j = 2, \ldots, \infty$" is fairly standard but it would be better as "$2 \leq j < \infty$"; we do not mean to suggest that $j = \infty$.

The equation in the footnote contains mistakes with the absolute value signs and parentheses. It should be:

$$\begin{aligned} \int_{\mathbb{R}^n} |g_{k,1}(\mathbf{x})||d^n\mathbf{x}| &= \int_{\mathbb{R}^n} \left|\sum_{i=1}^{\infty} f_{k,i}(\mathbf{x}) - \sum_{i=m(k)+1}^{\infty} f_{k,i}(\mathbf{x})\right| |d^n\mathbf{x}| \\ &\leq \int_{\mathbb{R}^n} \left|\sum_{i=1}^{\infty} f_{k,i}(\mathbf{x})\right||d^n\mathbf{x}| + \sum_{i=m(k)+1}^{\infty} \int |f_{k,i}(\mathbf{x})|\,|d^n\mathbf{x}| \\ &\leq \int_{\mathbb{R}^n} |f_k(\mathbf{x})||d^n\mathbf{x}| + \frac{1}{2^k}. \end{aligned}$$

Page 753 Two lines after Equation A21.49 replace "volume 0" by "measure 0" in two places. This uses Corollary A16.3, which has been corrected (it concerns measure, not volume).

Sentence right after Equation A21.50: third and fourth "equalities", not "inequalities."

Note: In line two of the proof, we are using Fubini for Riemann integrals. More precisely, Equation A21.49 is true for Riemann integrals if one ignores sets of measure 0, and so it is true without restriction for Lebesgue integrals.

Page 754 Third displayed equation: the bracket on the left should say "finite because f, hence g, is L-integrable."

In the paragraph beginning "For the converse", $\mathbb{R}^n$ should be $\mathbb{R}^{n+m}$; i.e., "every closed cube $C \in \mathcal{D}_0(\mathbb{R}^n)$ is compact" should be

"every closed cube $C \in \mathcal{D}_0(\mathbb{R}^{n+m})$ is compact."

Even with that correction, we were not quite rigorous in arguing that $f\chi_C$ is L-integrable. Here is another version:

Lemma *If f is L-integrable on $\mathbb{R}^n$, and g is an R-integrable function with $0 \le g \le 1$, then fg is L-integrable.*

Proof. Since f is L-integrable, we can set $f = \sum_k f_k$ with the f_k R-integrable and

$$\sum_k \int_{\mathbb{R}^n} |f_k(\mathbf{x})|\,|d^n\mathbf{x}| < \infty.$$

We have $fg = \sum_k f_k g$, where $f_k g$ is R-integrable; since $0 \le g \le 1$, we have

$$\sum_k \int_{\mathbb{R}^n} |f_k g(\mathbf{x})|\,|d^n\mathbf{x}| \le \sum_k \int_{\mathbb{R}^n} |f_k(\mathbf{x})|\,|d^n\mathbf{x}| < \infty.$$

Therefore fg is L-integrable. $\square$

Now take a closed cube $C \in \mathcal{D}_0(\mathbb{R}^n)$ and cover it by finitely many balls $B_1, B_2, \dots, B_N$, over which (by the first hypothesis of the converse) f is L-integrable. Then we can write

$$f\chi_C = f\chi_C\chi_{B_1} + f\chi_C\chi_{B_2 - \cup(B_2 - B1)} + \cdots + f\chi_C\chi_{B_N - \cup_{j=1}^{N-1} B_j}$$

By the above lemma, the terms on the right are all L-integrable, so by Proposition 4.11.15, $f\chi_C$ is L-integrable.

Page 755 We claim we are proving the "if" part in the text, leaving "if only" as an exercise. Actually, it's the reverse. It would be clearer to use $\Longrightarrow$ and $\Longleftarrow$. In the text we prove ($\Longrightarrow$) (that if f is L-integrable, then $|\det[\mathbf{D}\Phi]|(f\circ\Phi)$ is L-integrable and the formula is correct).

Margin note: The first and third equalities of Equation A21.59 are applications of Theorem 4.11.16. In both cases, the hypothesis of that theorem is satisfied by Equation A21.58. We could add an extra step:

$$\begin{aligned}\int_V f(\mathbf{v})|d^n\mathbf{v}| &\overbrace{=}^{\text{Eq.A21.57}} \int_V \sum_{k,i} f_{k,i}(\mathbf{v})|d^n\mathbf{v}| \overbrace{=}^{\text{Thm. 4.11.16}} \sum_{k,i}\int_V f_{k,i}(\mathbf{v})|d^n\mathbf{v}|\\ &\overbrace{=}^{\text{Thm. 4.10.12}} \sum_{k,i}\int_U \big|\det[\mathbf{D}\Phi(\mathbf{u})]\big|\; f_{k,i}(\Phi(\mathbf{u}))\;|d^n\mathbf{u}|\\ &\overbrace{=}^{\text{Thm. 4.11.16}} \int_U \big|\det[\mathbf{D}\Phi(\mathbf{u})]\big|\Big(\sum_{k,i} f_{k,i}(\Phi(\mathbf{u}))\Big)|d^n\mathbf{u}|;\end{aligned}$$

Page 755 Last line: "Exercise A21.2" should be "Exercise A21.5."

Page 759 Exercise A21.5 (last exercise of the section, incorrectly denoted A21.2): as indicated in the note for page 755, this exercise asks you to prove the "if" part, not "only if". In future editions this exercise will be:

Justify the ($\Longleftarrow$) part of Theorem 4.11.20 (if $|\det[\mathbf{D}\Phi]|(f \circ \Phi)$ is L-integrable, then f is L-integrable and the formula given in the theorem is correct), using the ($\Longrightarrow$) part and the chain rule.

Page 760 In Figure A22.1, the top lines in both rectangles should be darker.

Page 763 4th line: rather than state that the exterior derivative $d\varphi$ is a $(k+1)$-form, we should say "Since φ is a k-form, the exterior derivative $d\varphi$ should be a $(k+1)$-form. Thus we need to evaluate it on $k+1$ vectors and check that it is multilinear and alternating. This involves integrating φ ... "

Page 765 The margin note should start with "In," not "in."

Page 766 Definition A24.1 should read

Definition A24.1 (Pullback by a linear transformation). Let V, W be vector spaces, and $T : V \to W$ be a linear transformation. Then T^* is a linear transformation $A^k(W) \to A^k(V)$, defined as follows: if φ is a k-form on W, then $T^*\varphi$ is the k-form on V given by

$$T^*\varphi(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k) = \varphi\big(T(\vec{\mathbf{v}}_1), \dots, T(\vec{\mathbf{v}}_k)\big). \qquad \text{A24.1}$$

Page 767 In the last margin note, an end parenthesis is missing: $g(P_{\mathbf{f}(\mathbf{x})}$ should be $g(P_{\mathbf{f}(\mathbf{x})})$

In the line immediately before Definition A24.4 there is a superfluous comma.

Page 769 In the last line of Equation A24.14, the $\mathbf{g}^*\mathbf{f}^*$ should be $\mathbf{f}^*\mathbf{g}^*$:

$$= \mathbf{g}^*\varphi\Big(P_{\mathbf{f}(\mathbf{x})}\big([\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_1, \dots, [\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}_k\big)\Big) = \mathbf{f}^*\mathbf{g}^*\varphi\big(P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)\big)$$

Page 770 We have rewritten the first paragraph:

Why does this result matter? To define the exterior derivative, we used the parallelograms $P_{\mathbf{x}}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$. To do this, we had to know how to draw straight lines from one point to another; we were using the linear (straight) structure of a vector space. (We used $\mathbb{R}^n$, but any vector space would have done.) Theorem A24.8 says that "curved parallelograms" (little bits of manifolds) would have worked as well. Thus the exterior derivative is not restricted to forms defined on vector spaces.

(In this book we have discussed forms on vector spaces, but differential forms can also be defined on manifolds embedded in $\mathbb{R}^n$ and on abstract manifolds. Theorem A24.8 says that an exterior derivative exists for such forms. It is a crucial result, since forms without an exterior derivative would be of no interest.)

Title of Theorem A24.8: By "intrinsic" we mean "inherent: independent of some external conditions or circumstances." The pullback of a form by a C^1 mapping is a C^1 change of variables. Equation A24.17 says that when a form is pulled back by a C^1 mapping, its exterior derivative remains the same, translated appropriately into the new variables.

Page 770 In the first line of Equation A24.18 (last term), $[\mathbf{D}g(\mathbf{x})]$ should be $[\mathbf{Df}(\mathbf{x})]$:

$$\begin{aligned} \mathbf{f}^* dg\big(P_{\mathbf{x}}(\vec{\mathbf{v}})\big) &= dg\big(P_{\mathbf{f}(\mathbf{x})}[\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}}\big) = [\mathbf{D}g(\mathbf{f}(\mathbf{x}))][\mathbf{Df}(\mathbf{x})]\vec{\mathbf{v}} \\ &= [\mathbf{D}g \circ \mathbf{f}(\mathbf{x})]\vec{\mathbf{v}} = d(g\circ \mathbf{f})\big(P_{\mathbf{x}}(\vec{\mathbf{v}})\big) = d(\mathbf{f}^* g)\big(P_{\mathbf{x}}(\vec{\mathbf{v}})\big). \end{aligned} \qquad \text{A24.18}$$

Equation A24.19: above the first equal sign, "Theorem A6.7.8" should be "Theorem 6.7.8."

Page 782 In Exercise A25.2, "(proof of Lemma A25.12)" should be "(see Equation A25.12)".

Inside back cover The "useful formulas: trigonometry" would be more useful if they were all correct! Sorry! The fourth and fifth formulas should be

$$\cos\alpha = \sin(\pi/2 - \alpha) \quad \text{and} \quad \sin\alpha = \cos(\pi/2-\alpha).$$

Index

Page 792 dominated convergence (Lebesgue), 515 (not 516)
The listing for *diffeomorphism* on page 514 should be deleted.

Page 797 triangle inequality, 76–77 (not 76)

Additions to index: