Statement and proof of a strong form of Poincaré's lemma

Theorem 6.12.12 (Poincaré's lemma: a variant). *Let $U \subset \mathbb{R}^n$ be open and star-shaped with respect to $\mathbf{0}$. Let φ be a k-form on U. Then*

$$\varphi = \mathbf{d}(\mathbf{c}\varphi) + \mathbf{c}(\mathbf{d}\varphi). \qquad 6.12.29$$

In particular, if $\mathbf{d}\varphi = 0$, then $\varphi = \mathbf{d}(\mathbf{c}\varphi)$.

Exercise 6.12.9 asks you to rewrite this proof with the proper detail and to justify the last equality.

Proof. Written without all the proper detail, we have

$$\mathbf{d}(\mathbf{c}\varphi)(P) \overset{1}{=} (\lim) \int_{\partial P_h} \mathbf{c}\varphi \overset{2}{=} (\lim) \int_{C\partial P_h} \varphi \overset{3}{=} (\lim) \int_{P_h} \varphi - (\lim) \int_{\partial C P_h} \varphi$$
$$\overset{4}{=} (\lim) \int_{P_h} \varphi - (\lim) \int_{C P_h} \mathbf{d}\varphi = \varphi(P) - \mathbf{c}(\mathbf{d}\varphi)(P). \qquad 6.12.30$$

Equality (1) is the definition of the exterior derivative, (2) is the definition of the cone operator, (3) is equation 6.12.13, and (4) is Stokes's theorem. □

Example 6.12.13 (Poincaré's lemma). Let $\varphi = x_3\, dx_1 \wedge dx_3$, as in example 6.12.11. Note that $\mathbf{d}(x_3\, dx_1 \wedge dx_3) = 0$. Therefore, by theorem 6.12.12, we should have

$$\underbrace{\mathbf{d}(\mathbf{c}\, x_3\, dx_1 \wedge dx_3)}_{\mathbf{dc}\varphi} = \underbrace{x_3\, dx_1 \wedge dx_3}_{\varphi}. \qquad 6.12.31$$

Indeed,

$$\mathbf{d}(\mathbf{c}x_3\, dx_1 \wedge dx_3) \underbrace{=}_{\text{Eq. 6.12.28}} \mathbf{d}\left(-\frac{x_3^2}{3}\, dx_1 + \frac{x_1 x_3}{3}\, dx_3\right)$$
$$= -\frac{2x_3}{3}\, dx_3 \wedge dx_1 + \frac{x_3}{3}\, dx_1 \wedge dx_3 \qquad 6.12.32$$
$$= \frac{2}{3} x_3\, dx_1 \wedge dx_3 + \frac{1}{3} x_3\, dx_1 \wedge dx_3 = x_3\, dx_1 \wedge dx_3. \quad \triangle$$

Potentials and electromagnetism

A force field $\vec{F}$ is conservative if $\mathbf{d}W_{\vec{F}} = 0$. In some sense

$$\mathbf{d}\mathbb{F} = 0$$

says that electromagnetism is conservative.

Spacetime is of course convex.

Recall that one way of stating Maxwell's equations is (equation 6.11.15)

$$\mathbf{d}\mathbb{F} = 0, \qquad \mathbf{d}\mathbb{M} = 4\pi\mathbb{J}. \qquad 6.12.33$$

From the first of these equation and the Poincaré lemma, it follows that there is a 1-form $\mathbb{A}$ on spacetime such that $\mathbf{d}\mathbb{A} = \mathbb{F}$, i.e., $\mathbb{A}$ is a potential for $\mathbb{F}$.

In any splitting of spacetime into space and time, any 1-form on spacetime can be written

$$\mathbb{A} = \frac{1}{c} W_{\vec{A}} - V c\, dt, \qquad 6.12.34$$

where $W_{\vec{A}}$ is the work of some vector field $\vec{A}$ (called the *vector potential*) and V is a function (called the *scalar potential*). Then, since

$$\mathbf{d}\mathbb{A} = \frac{1}{c}\Phi_{\vec{\nabla}\times\vec{A}} - \frac{1}{c^2}W_{D_t\vec{A}} \wedge cdt - W_{\vec{\nabla}V} \wedge cdt = \mathbb{F}, \qquad 6.12.35$$

by $\mathbb{F} = W_{\vec{\mathbf{E}}} \wedge cdt + \Phi_{\vec{\mathbf{B}}}$ (equation 6.11.8) we have

$$\vec{\mathbf{E}} = -(\vec{\nabla}V + \frac{1}{c^2}D_t\vec{A}) \quad \text{and} \quad \vec{\mathbf{B}} = \frac{1}{c}\vec{\nabla}\times\vec{A}, \qquad 6.12.36$$

so that $\mathbb{M} \stackrel{\text{def}}{=} W_{\vec{\mathbf{B}}} \wedge cdt - \Phi_{\vec{\mathbf{E}}}$ (the second equation in 6.11.8) becomes

$$\mathbb{M} = W_{\frac{1}{c}\vec{\nabla}\times\vec{A}} \wedge cdt + \Phi_{(\vec{\nabla}V + \frac{1}{c^2}D_t\vec{A})}. \qquad 6.12.37$$

Since $\mathbf{d}\mathbb{F} = 0$ has been built into the formula, the equation $\mathbf{d}\mathbb{M} = 4\pi\mathbb{J}$ encodes all of electromagnetism. It becomes

$$\begin{aligned}\mathbf{d}\mathbb{M} &= \Phi_{(\frac{1}{c}\vec{\nabla}\times(\vec{\nabla}\times\vec{A})) + \frac{1}{c^3}(D_t^2\vec{A} + \frac{1}{c}\vec{\nabla}D_tV)} \wedge cdt + M_{\vec{\nabla}\cdot(\frac{1}{c^2}D_t\vec{A}+\vec{\nabla}V)} \\ &= 4\pi\Big(\frac{1}{c}\Phi_{\vec{\mathbf{j}}} \wedge c\,dt - M_\rho\Big).\end{aligned} \qquad 6.12.38$$

Written in components, changing sign and multiplying through by c for the first equation, this becomes

$$-\vec{\nabla}\times(\vec{\nabla}\times\vec{A}) - \left(\frac{1}{c^2}D_t^2\vec{A} + \vec{\nabla}D_tV\right) = -4\pi\vec{\mathbf{j}} \qquad 6.12.39$$

$$\vec{\nabla}\cdot\left(\frac{1}{c^2}D_t\vec{A} + \vec{\nabla}V\right) = -4\pi\rho. \qquad 6.12.40$$

A first way to improve these equations is to recall (equation 6.11.64) that the Laplacian $\vec{\Delta}\vec{A}$ of a vector field $\vec{A}$ is $\vec{\Delta}\vec{A} = \operatorname{grad}\operatorname{div}\vec{A} - \operatorname{curl}\operatorname{curl}\vec{A}$, and the Laplacian ΔV of a function V is $\Delta V = \operatorname{div}\operatorname{grad} V$. Using these relations, we can rewrite equations 6.12.39 and 6.12.40 as

$$(\vec{\Delta}\vec{A} - \frac{1}{c^2}D_t^2\vec{A}) - \vec{\nabla}(\vec{\nabla}\cdot\vec{A} + D_tV) = -4\pi\vec{\mathbf{j}} \qquad 6.12.41$$

$$\Delta V + \vec{\nabla}D_t\vec{A} = -4\pi\rho. \qquad 6.12.42$$

This system of equations, although still scary, is much better than Maxwell's equation: it is an equation for four unknown functions instead of six. But they can be simplified further. Recall that $\mathbb{A}$ is only defined up to addition of $\mathbf{d}f$ for some function f on spacetime. It turns out that one can choose f so that after adding $\frac{1}{c}\mathbf{d}f$ to our potential $\mathbb{A}$, the new potential will satisfy the *Lorenz gauge condition*

$$\vec{\nabla}\cdot\vec{A} + D_tV = 0. \qquad 6.12.43$$

Using this identity, our system of equations becomes

$$\vec{\Delta}\vec{A} - \frac{1}{c^2}D_t^2\vec{A} = -4\pi\vec{\mathbf{j}} \qquad 6.12.44$$

$$\Delta V - \frac{1}{c^2}D_t^2 V = -4\pi\rho. \qquad 6.12.45$$

The choice of scalar potential and vector potential depend, of course, on the splitting of spacetime into space and time. In addition, if $\mathbf{d}\mathbb{A} = \mathbb{F}$, then for any function f we have $\mathbf{d}(\mathbb{A} + df) = \mathbb{F}$.

Second line of equation 6.12.38: Recall (equation 6.11.12) that

$$\mathbb{J} = \frac{1}{c}\Phi_{\vec{\mathbf{j}}} \wedge c\,dt - M_\rho$$

A potential $\mathbb{A}$ is only defined up to adding $\mathbf{d}f$ for an arbitrary function f. It seems that this should mean that $\mathbb{A}$ is a mathematical construct without physical reality. This is not true: Hermann Weyl, in 1928, found that $\mathbb{A}$ can be understood as a "connection in a 1-dimensional bundle"; just as gravity is the curvature of spacetime in general relativity, the curvature of the connection $\mathbb{A}$ is the electromagnetic field. Adding $\mathbf{d}f$ means "using different coordinates in the bundle."

This idea turned out to be immensely important: in 1954 Yang and Mills interpreted the strong nuclear force, which keeps atomic nuclei together, as the curvature of a connection in a 2-dimensional bundle. Soon thereafter, gauge theory took over particle physics completely; today particle physics is gauge theory.

In particular particle physicists are now differential geometers, and the interaction between these branches of mathematics and physics has been immensely profitable to both fields. Simon Donaldson, Michael Freedman, and Edward Witten each received the Fields Medal for work in gauge field theory.