

Since  $f_n(X) \subset S$  and  $S$  is closed in  $(Y, \rho)$ , equation (D.6.1) implies that

$$g(x) = \lim_{n \rightarrow \infty} f_n(x) \in \overline{S} = S \quad \text{for all } x \in X.$$

Hence,  $g(X) \subset S$  so that  $g \in M$ . So,  $M$  is closed in  $C(X, Y)$ . Because  $C(X, Y)$  is complete,  $M$  must also be complete (by Proposition 1.8.8).

**6.3.9** Let  $y_1(x) := e^{\alpha x} \cos \beta x$ . Then

$$\begin{aligned} y_1'(x) &= e^{\alpha x}(\alpha \cos \beta x - \beta \sin \beta x), \\ y_1''(x) &= e^{\alpha x}((\alpha^2 - \beta^2) \cos \beta x - 2\alpha\beta \sin \beta x). \end{aligned} \tag{D.6.2}$$

Hence,

$$y_1''(x) + by_1'(x) + cy_1(x) = e^{\alpha x}((\alpha^2 - \beta^2 + b\alpha + c) \cos \beta x - (2\alpha\beta + b\beta) \sin \beta x).$$

Since  $b^2 - 4c < 0$ , we have  $4c - b^2 > 0$  and the quadratic formula yields

$$\alpha = \frac{-b}{2} \quad \text{and} \quad \beta = \frac{\sqrt{4c - b^2}}{2}.$$

Hence,

$$\begin{aligned} \alpha^2 - \beta^2 + b\alpha + c &= \frac{b^2}{4} + \frac{b^2 - 4c}{4} - \frac{b^2}{2} + c = 0 \\ 2\alpha\beta + b\beta &= \frac{-b\sqrt{4c - b^2}}{2} + \frac{b\sqrt{4c - b^2}}{2} = 0. \end{aligned}$$

Thus, (D.6.2) implies that  $y_1$  is a solution to  $y'' + by' + cy = 0$ . Similarly, let  $y_2(x) := e^{\alpha x} \sin \beta x$ . Then

$$\begin{aligned} y_2'(x) &= e^{\alpha x}(\alpha \sin \beta x + \beta \cos \beta x) \\ y_2''(x) &= e^{\alpha x}((\alpha^2 - \beta^2) \sin \beta x + 2\alpha\beta \cos \beta x), \end{aligned}$$

so

$$y_2''(x) + by_2'(x) + cy_2(x) = e^{\alpha x}((\alpha^2 - \beta^2 + b\alpha + c) \sin \beta x + (2\alpha\beta + b\beta) \cos \beta x) = 0.$$

Thus,  $y_2$  is also a solution to  $y'' + by' + cy = 0$ . Finally, suppose that  $c_1 y_1 + c_2 y_2 = \mathbf{0}$  for some real numbers  $c_1, c_2$ . Then  $c_1 \cos \beta x + c_2 \sin \beta x = 0$  for all  $x \in \mathbb{R}$ . Substituting  $x := 0$  gives  $c_1 = 0$ . Thus,  $c_2 = 0$  also. Hence,  $\{y_1, y_2\}$  is linearly independent. Thus, by Theorem 6.3.28,  $\{y_1, y_2\}$  forms a basis for the solutions space of  $y'' + by' + cy = 0$ .

**6.3.11** 1. The proof will be by induction on  $n$ . For  $n = 1$ , this is trivial. Assume that whenever  $r_1, \dots, r_n$  are distinct real numbers, then  $\{E_{r_1}, \dots, E_{r_n}\}$  is linearly independent. Let  $b_1, \dots, b_{n+1}$  be distinct real numbers. Denote the zero function on  $\mathbb{R}$  and the constant function  $x \mapsto 1$  on  $\mathbb{R}$  by  $\mathbf{0}$  and  $\mathbf{1}$ , respectively. Suppose that  $c_1, \dots, c_n, c_{n+1}$  are real numbers such that

$$c_1 E_{b_1} + \dots + c_n E_{b_n} + c_{n+1} E_{b_{n+1}} = \mathbf{0}. \tag{D.6.3}$$

Put  $r_k := b_k - b_{n+1}$  for  $1 \leq k \leq n$ . When we divide by  $E_{b_{n+1}}$ , equation (D.6.3) becomes

$$c_1 E_{r_1} + \dots + c_n E_{r_n} = -c_{n+1} \mathbf{1}. \tag{D.6.4}$$

Thus, taking derivative in both sides of (D.6.4), we obtain

$$c_1 r_1 E_{r_1} + \cdots + c_n r_n E_{r_n} = \mathbf{0}.$$

Now,  $r_1, \dots, r_n$  are all distinct and nonzero (because  $b_1, \dots, b_{n+1}$  are all distinct), the inductive hypothesis implies that  $c_1 r_1 = \cdots = c_n r_n = 0$ . Hence,  $c_1 = \cdots = c_n = 0$ . It now follows from (D.6.3) that  $c_{n+1} = 0$  also. Hence,  $\{E_{b_1}, \dots, E_{b_n}, E_{b_{n+1}}\}$  is linearly independent.

2. We will show that each  $E_{r_k}$  is a solution to (6.3.81) on  $(-\infty, \infty)$ . For each  $1 \leq \ell, k \leq n$ , we have

$$E_{r_k}^{(\ell)} = r_k^\ell E_{r_k}.$$

Hence, each  $E_{r_k}$  is a solution to (6.3.81), since

$$\begin{aligned} E_{r_k}^{(n)} + a_{n-1} E_{r_k}^{(n-1)} + \cdots + a_1 E_{r_k}' + a_0 E_{r_k} \\ = r_k^n E_{r_k} + a_{n-1} r_k^{n-1} E_{r_k} + \cdots + a_1 r_k E_{r_k} + a_0 E_{r_k} \\ = (r_k^n + a_{n-1} r_k^{n-1} + \cdots + a_1 r_k + a_0) E_{r_k} = \mathbf{0}. \end{aligned}$$

Thus, by part 1, we now know that  $\{E_{r_1}, \dots, E_{r_n}\}$  is a linearly independent set of  $n$  solutions to (6.3.81). Hence, by Theorem 6.3.28,  $\{E_{r_1}, \dots, E_{r_n}\}$  is a basis for the solutions space of (6.3.81). The desired conclusion follows.

**6.3.13** For each integer  $1 \leq k \leq n$ , let  $f_k : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}$  be defined by

$$f_k \left( t, \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \right) := e^{t^2} \frac{x_k}{1 + x_1^2 + \cdots + x_n^2}.$$

Then for any  $t \in \mathbb{R}$ , the function  $\frac{\partial f_k}{\partial x_j}(t, \cdot)$  is continuous on  $\mathbb{R}^n$  for all  $1 \leq k, j \leq n$ . So, by Lemma 6.3.10, each  $f_k$  is Lipschitz with respect to the variable in  $\mathbb{R}^n$ . Put

$\mathbf{f} := \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$ . Then by Remark 6.3.9,  $\mathbf{f}$  is Lipschitz with respect to the variable in

$\mathbb{R}^n$ . Now, apply Theorem 6.3.12 with  $t_0 := 0$  and  $\mathbf{z}_0 := \begin{pmatrix} 1 \\ \vdots \\ n \end{pmatrix}$ .

**6.3.15** No. Let  $a_{m-1}, \dots, a_1, a_0$  be as in the statement of the problem. For any continuous  $h : (-1, 1) \rightarrow (-1, 1)$ , consider the differential equation

$$y^{(m)} + a_{m-1} y^{(m-1)} + \cdots + a_1 y' + a_0 y = h. \quad (\text{D.6.5})$$

Suppose on the contrary that there are  $f_1$  and  $f_2$  satisfying (D.6.5). Then  $f_1 - f_2$  is a solution to the homogeneous differential equation

$$y^{(m)} + a_{m-1} y^{(m-1)} + \cdots + a_1 y' + a_0 y = 0. \quad (\text{D.6.6})$$

By assumption, the equation  $r^m + a_{m-1} r^{m-1} + \cdots + a_1 r + a_0 = 0$  has  $m$  distinct real roots. Hence, by Exercise 6.3.11, the solutions set to (D.6.6) has a basis consisting of the functions  $E_{r_j} : t \mapsto e^{r_j t}, 1 \leq j \leq m$ . Thus,  $f_1 - f_2$  can be written as a linear combination of  $E_{r_j}, 1 \leq j \leq m$ . This is a contradiction, since

$$f_1(t) - f_2(t) = \cos^2 t - \sin^2 t = \cos(2t) \quad \text{for all } t \in (-1, 1).$$